PDE Introductory Exercises and Solutions
Chapter 5, Hyperbolic Equations

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Contents

1 Derive the wave equation for a string that moves in a medium...

Derive the wave equation for a string that moves in a medium in which the resistance force between the string and the medium is proportional to the velocity of the string.

Solution. Since the resistance force between the string and the medium is proportional to the velocity of the string, take the resistance on a unit length of string as $R = -\beta u_t$ where β is the positive damping coefficient.

Applying the string micro-piece model with the tension $T(t)$ of the string and the external force $F(x,t)$ exerted on the string

$$\int_{x_1}^{x_2} \rho(x)u_{tt}(x,t)\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial\left(T(t)u_x(x,t)\right)}{\partial x}\mathrm{d}x + \int_{x_1}^{x_2} -\beta u_t(x,t)\mathrm{d}x + \int_{x_1}^{x_2} F(x,t)\mathrm{d}x.$$

It is not hard to deduce the equation

$$\rho(x)u_{tt}(x,t) = T(t)u_{xx}(x,t) - \beta u_t(x,t) + F(x,t),$$

then, dividing by given constant $\rho(x)$ as density distribution of the string, we have

$$u_{tt} = a^2 u_{xx} - bu_t + f(x,t), \quad x \in (0,l),\ t \in (-\infty,+\infty)$$

where $b = \beta/\rho(x)$, $a^2 = T(t)/\rho(x)$ and $f(x,t) = F(x,t)/\rho(x)$.

2 Solve the following initial-boundary value problems for the wave equation.

(1)

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \quad x \in (0,1),\ t \in \mathbb{R}, \\ u(0,t) = 0 = u(1,t), \quad t \in \mathbb{R}, \\ u(x,0) = \sin(\pi x),\ u_t(x,0) = \sin(4\pi x), \quad x \in (0,1). \end{cases}$$

Solution. Assume $u(x,t) = X(x)T(t)$,

$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda.$$

The corresponding eigenvalue problem of X is

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0,\ x(1) = 0 \end{cases}$$

then the eigenvalue and eigenfunction are given by

$$\lambda_n = n^2\pi^2, \quad n = 1, 2, \cdots,$$

$$X_n(x) = \sin(n\pi x), \quad n = 1, 2, \cdots.$$

The corresponding ODE of T is $T_n''(t) + 4\lambda_n T_n(t) = 0$ and the general solution is given by

$$T_n(t) = a_n \cos(2n\pi t) + b_n \sin(2n\pi t), \quad n = 1, 2, \cdots.$$

By the initial condition $u(x, 0) = \sin(\pi x)$ and $T_n(0) = a_n$,

$$\sin(\pi x) = \sum_{n=1}^{\infty} X_n(x) T_n(0) = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot a_n \quad \Rightarrow a_1 = 1,\ a_{n \geq 2} = 0,$$

and by the initial condition $u_t(x, 0) = \sin(4\pi x)$ and $T_n'(0) = 2n\pi b_n$,

$$\sin(4\pi x) = \sum_{n=1}^{\infty} X_n(x) T_n'(0) = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot 2n\pi b_n \quad \Rightarrow b_4 = \frac{1}{8\pi},\ b_{n \neq 4} = 0.$$

Thus,

$$u(x, t) = \cos(2\pi t) \sin(\pi x) + \frac{1}{8\pi} \sin(8\pi t) \sin(4\pi x).$$

(2)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in (0, 1),\ t \in \mathbb{R}, \\ u_x(0, t) = 0 = u_x(1, t), \quad t \in \mathbb{R}, \\ u(x, 0) = x,\ u_t(x, 0) = 0, \quad x \in (0, 1). \end{cases}$$

Solution. Assume $u(x, t) = X(x)T(t)$,

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda.$$

The corresponding eigenvalue problem of X is

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0,\ x(1) = 0 \end{cases}$$

then the eigenvalue and eigenfunction are given by

$$\lambda_n = n^2 \pi^2, \quad n = 0, 1, 2, \cdots,$$

$$X_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, \cdots.$$

The corresponding ODE of T is $T_n''(t) + a^2 \lambda_n T_n(t) = 0$ and the general solution is given by

$$T_0(t) = a_0 + b_0 t,$$

$$T_n(t) = a_n \cos(an\pi t) + b_n \sin(an\pi t), \quad n = 1, 2, \cdots.$$

By the initial condition $u(x, 0) = x$,

$$x = \sum_{n=0}^{\infty} X_n(x) T_n(0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

$$
\begin{aligned}
a_0 &= \int_0^1 x \mathrm{d}x = \frac{1}{2}, \\
a_n \cos(n\pi) &= \frac{2}{1}\int_0^1 x\cos(n\pi x)\mathrm{d}x = 2\frac{\pi n \sin(\pi n) + \cos(\pi n) - 1}{\pi^2 n^2} \\
\Rightarrow a_n &= \frac{2(-1)^n - 2}{n^2\pi^2}, \quad n = 1, 2, \cdots,
\end{aligned}
$$

and by the initial condition $u_t(x, 0) = 0$,

$$
0 = \sum_{n=0}^{\infty} X_n(x)T_n'(0) = b_0 + \sum_{n=1}^{\infty} b_n a n\pi \cos(n\pi x) \quad \Rightarrow b_n = 0, \quad n \geq 0.
$$

Thus,

$$
u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 2}{n^2\pi^2} \cos(an\pi t)\cos(n\pi x).
$$

(3)

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0, \quad x \in (0,1),\ t \in \mathbb{R}, \\
u(0,t) = 0,\ u_x(1,t) = 1, \quad t \in \mathbb{R}, \\
u(x,0) = 0,\ u_t(x,0) = \cos(\pi x), \quad x \in (0,1).
\end{cases}
$$

Solution. To transform non-homogeneous boundary conditions into a homogeneous, let

$$
u(x,t) = U(x,t) + w(x,t) \quad \Rightarrow U(x,t) = u(x,t) - w(x,t)
$$

and we should choose $w(x,t)$ for the following

$$
U(0,t) = -w(0,t), \quad U_x(0,t) = u_x(1,t) - w_x(1,t) = 1 - w_x(1,t)
$$

s.t. each of them equals to zero. Here, we may take $w(x,t) = x$ so $U(x,t) = u(x,t) - x$, then there is

$$
\begin{cases}
U_{tt} - a^2 U_{xx} = 0, \quad x \in (0,1),\ t \in \mathbb{R}, \\
U(0,t) = 0,\ U_x(1,t) = 0, \quad t \in \mathbb{R}, \\
U(x,0) = -x,\ U_t(x,0) = \cos(\pi x), \quad x \in (0,1).
\end{cases}
$$

To solve this problem, we can start the routine now. Assume $U(x,t) = X(x)T(t)$,

$$
\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda.
$$

The corresponding eigenvalue problem of X is

$$
\begin{cases}
X''(x) + \lambda X(x) = 0 \\
X(0) = 0,\ X'(1) = 0
\end{cases}
$$

then the eigenvalue and eigenfunction are given by

$$
\lambda_n = \left(n\pi + \frac{\pi}{2}\right)^2, \quad n = 0, 1, 2, \cdots,
$$

$$X_n(x) = \sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right), \quad n = 0, 1, 2, \cdots.$$

The corresponding ODE of T is $T_n''(t) + a^2\lambda_n T_n(t) = 0$ and the general solution is given by

$$T_n(t) = a_n \cos\left(a\left(n\pi + \frac{\pi}{2}\right)t\right) + b_n \sin\left(a\left(n\pi + \frac{\pi}{2}\right)t\right), \quad n = 0, 1, 2, \cdots.$$

By the initial condition $U(x, 0) = x$,

$$-x = \sum_{n=0}^{\infty} X_n(x)T_n(0) = \sum_{n=0}^{\infty} a_n \sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right)$$

$$\Rightarrow a_n = -2\int_0^1 x \sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right)\mathrm{d}x = (-1)^{n+1}\frac{2}{(n+1/2)^2\pi^2}, \quad n = 0, 1, 2, \cdots.$$

and by the initial condition $U_t(x, 0) = \cos(\pi x)$,

$$\cos \pi x = \sum_{n=0}^{\infty} b_n a(n + 1/2)\pi \sin\left(n\pi + \frac{\pi}{2}\right)x$$

$$\begin{aligned}\Rightarrow b_n &= \frac{2}{a(n+1/2)\pi}\int_0^1 \cos(\pi x)\sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right)\mathrm{d}x \\ &= \frac{8}{a\pi^2(4n^2 + 4n - 3)}, \quad n = 0, 1, 2, \cdots.\end{aligned}$$

Thus,

$$U(x,t) = \sum_{n=0}^{\infty}\left(a_n \cos\left(a\left(n\pi + \frac{\pi}{2}\right)t\right) + b_n \sin\left(a\left(n\pi + \frac{\pi}{2}\right)t\right)\right)\cdot \sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right)$$

so that

$$\begin{aligned}u(x,t) = \sum_{n=0}^{\infty}\Bigg(\left((-1)^{n+1}\frac{2}{(n+1/2)^2\pi^2}\right)\cos\left(a\left(n\pi + \frac{\pi}{2}\right)t\right) \\ + \left(\frac{8}{a\pi^2(4n^2+4n-3)}\right)\sin\left(a\left(n\pi + \frac{\pi}{2}\right)t\right)\Bigg)\cdot \sin\left(\left(n\pi + \frac{\pi}{2}\right)x\right) + x.\end{aligned}$$

3 Solve the Cauchy problem.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = e^{-x^2},\ u_t(x,0) = \sin(x), \quad x \in \mathbb{R}. \end{cases}$$

Solution. By d'Alembert's Formula, with $\phi(x) = e^{-x^2}$ and $\psi(x) = \sin(x)$,

$$\begin{aligned}u(x,t) &= \frac{1}{2}\left(\phi(x+at) + \phi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s \\ &= \frac{1}{2}\left(e^{-(x-at)^2} + e^{-(x+at)^2}\right) + \frac{1}{2a}\int_{x-at}^{x+at}\sin(s)\mathrm{d}s \\ &= \frac{1}{2}\left(e^{-(x-at)^2} + e^{-(x+at)^2}\right) + \frac{1}{2a}\left(\cos(x-at) - \cos(x+at)\right).\end{aligned}$$

4 Solve the Cauchy problem.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = 0,\ u_t(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where

$$\psi(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| \geq a. \end{cases}$$

Sketch the graph of u vs x at times $t = 1/2,\ 1,\ 3/2,\ 2$.

Solution. By d'Alembert's Formula, with $\phi(x) = 0$,

$$u(x,t) = \frac{1}{2}\left(\phi(x+at) + \phi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\mathrm{d}s$$

$$= \begin{cases} 0, & x - at \geq a \\ \frac{1}{2a}\int_{x-at}^{a} \mathrm{d}s, & -a < x - at < a,\ a < x + at, \\ \frac{1}{2a}\int_{x-at}^{x+at} \mathrm{d}s, & -a < x - at,\ x + at < a, \\ \frac{1}{2a}\int_{-a}^{a} \mathrm{d}s, & x - at < -a,\ a < x + at, \\ \frac{1}{2a}\int_{-a}^{x+at} \mathrm{d}s, & x - at < -a,\ -a < x + at < a, \\ 0, & x + at \leq -a \end{cases}$$

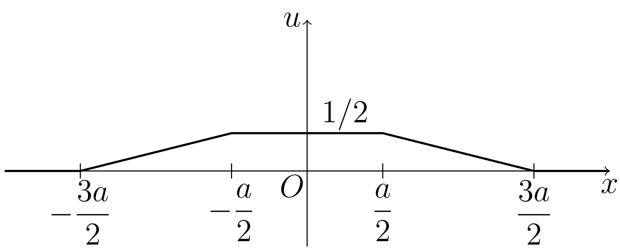
That is,

$$u(x,t) = \begin{cases} 0, & x - at \geq a \text{ or } x + at \leq -a, \\ \frac{a - x + at}{2a}, & -a < x - at < a,\ a < x + at, \\ t, & -a \leq x - at,\ x + at \leq a, \\ 1, & x - at \leq -a,\ a \leq x + at, \\ \frac{x + at + a}{2a}, & x - at < -a,\ -a < x + at < a. \end{cases}$$

The graphs of u vs x:

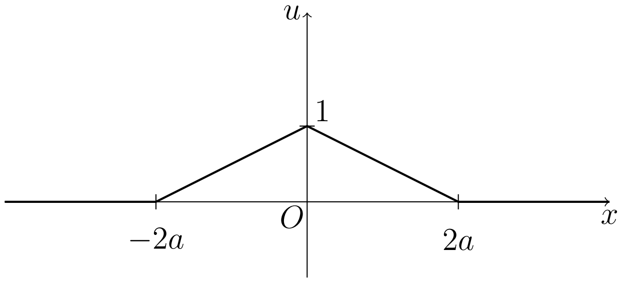
If $t = 1/2$, then there is

$$u(x,t) = \begin{cases} 0, & x \geq 3a/2 \text{ or } x \leq -3a/2, \\ \frac{-x + 3a/2}{2a}, & a/2 < x < 3a/2, \\ 1/2, & -a/2 \leq x \leq a/2, \\ \frac{x + 3a/2}{2a}, & -3a/2 < x < -a/2. \end{cases}$$



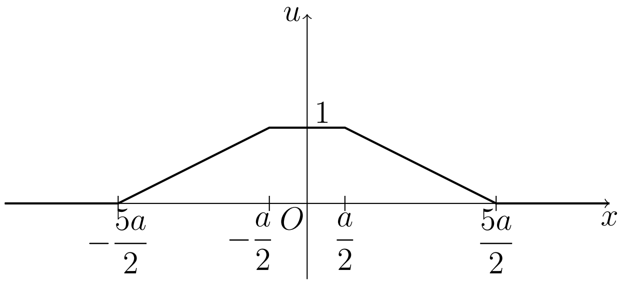
If $t = 1$, then there is

$$u(x,t) = \begin{cases} 0, & x \geq 2a \text{ or } x \leq -2a, \\ \dfrac{-x+2a}{2a}, & 0 < x < 2a, \\ 1, & x = 0 \\ \dfrac{x+2a}{2a}, & -2a < x < 0. \end{cases}$$



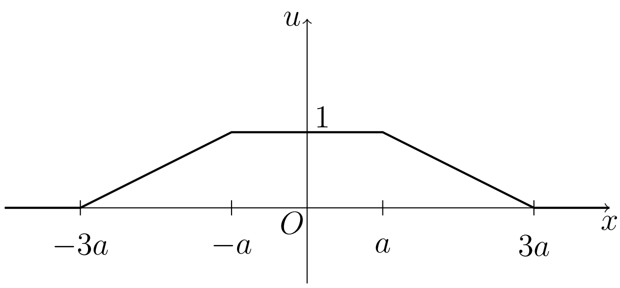
If $t = 3/2$, then there is

$$u(x,t) = \begin{cases} 0, & x \geq 5a/2 \text{ or } x \leq -5a/2, \\ \dfrac{-x+5a/2}{2a}, & a/2 < x < 5a/2, \\ 1, & -a/2 \leq x \leq a/2, \\ \dfrac{x+5a/2}{2a}, & -5a/2 < x < -a/2. \end{cases}$$



If $t = 2$, then there is

$$u(x,t) = \begin{cases} 0, & x \geq 3a \text{ or } x \leq -3a, \\ \dfrac{-x+3a}{2a}, & a < x < 3a, \\ 1, & -a \leq x \leq a, \\ \dfrac{x+3a}{2a}, & -3a < x < -a. \end{cases}$$



5 Let u be a solution of the wave equation...Is it possible that $u(x,1)$ is smoother than $u(x,0)$? Is it possible to have a maximum principle for the wave equation?

Let u be a solution of the wave equation

$$u_{tt} = u_{xx}, \quad x \in \mathbb{R},\ t \in \mathbb{R}.$$

Is it possible that $u(x,1)$ is smoother than $u(x,0)$? Is it possible to have a maximum principle for the wave equation?

Solution.
(a) It is impossible that $u(x,1)$ is smoother than $u(x,0)$ in x.
Denoted the initial conditions as $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, consider

$$u_x(x,0) = \frac{\partial}{\partial x}\left(\frac{1}{2}\left(\phi(x)+\phi(x)\right) + \frac{1}{2}\int_x^x \psi(s)\mathrm{d}s\right) = \frac{\mathrm{d}\phi(x)}{\mathrm{d}x}, \quad u_{xx}(x,0) = \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2}$$

then

$$\begin{aligned} u_x(x,1) &= \frac{\partial}{\partial x}\left(\frac{1}{2}\left(\phi(x+1)+\phi(x-1)\right) + \frac{1}{2}\int_{x-1}^{x+1} \psi(s)\mathrm{d}s\right) \\ &= \frac{1}{2}\frac{\mathrm{d}\phi(x+1)}{\mathrm{d}x} + \frac{1}{2}\frac{\mathrm{d}\phi(x-1)}{\mathrm{d}x} + \frac{1}{2}\frac{\partial\left(\int_{x-1}^{x+1}\psi(s)\mathrm{d}s\right)}{\partial x}, \\ u_{xx}(x,1) &= \frac{1}{2}\frac{\mathrm{d}^2\phi(x+1)}{\mathrm{d}x^2} + \frac{1}{2}\frac{\mathrm{d}^2\phi(x-1)}{\mathrm{d}x^2} + \frac{1}{2}\frac{\partial^2\left(\int_{x-1}^{x+1}\psi(s)\mathrm{d}s\right)}{\partial x^2}. \end{aligned}$$

And the derivatives of higher degrees are similar. We have that $u(x,0)$ is at least smooth as $u(x,1)$, because (for all $x \in \mathbb{R}$) the existence of $\dfrac{\mathrm{d}^k\phi(x+1)}{\mathrm{d}x^k}$, $k = 1,2,\cdots$ promises the existence of $\dfrac{\mathrm{d}^k\phi(x)}{\mathrm{d}x^k}$, $k = 1,2,\cdots$.

(b) There is no a maximum principle for the wave equation. Construct a counterexample as following:

Suppose that there is a maximum principle for the wave equation and $u(x,t) = \sin x \sin t$ is a solution for the wave equation in a rectangle $(0,L)\times(0,T) := (0,\pi)\times(0,\pi)$, where $u_{xx} = -\sin x \sin t = u_{tt}$. According to the maximum principle, the maximum of $u(x,t)$ is zero since

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = 0, \quad u(x,\pi) = 0,$$

but in the rectangle there is $u\left(\dfrac{\pi}{2},\dfrac{\pi}{2}\right) = 1$, which is a contradiction.

Besides, we also see that a solution for the wave equation may not satisfy the prerequisite $-\Delta u \le 0$, for above example

$$-\Delta u = -u_{xx} - u_{tt} = 2\sin x \sin t \nleq 0.$$

6 Let u be the solution of... Show that if both ϕ and ψ are even functions, then so is u in x. Formulate and prove the analog when both ϕ and ψ are odd.

Let u be the solution of

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases}$$

Show that if both ϕ and ψ are even functions, then so is u in x. Formulate and prove the analog when both ϕ and ψ are odd.

(1) For Even Initial Conditions

We want to show that if both ϕ and ψ are even functions, then so is u in x

Even $\phi(x)$ and $\psi(x)$ give that $\phi(x) = \phi(-x)$ and $\psi(x) = \psi(-x)$. The associated initial value problem for $u(-x,t)$ is

$$\begin{cases} u_{tt}(-x,t) - a^2 u_{xx}(-x,t) = 0, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(-x,0) = \phi(-x) = \phi(x),\ u_t(-x,0) = \psi(-x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

and consider $u(x,t) - u(-x,t)$ as a linear combination of two solution of the wave equation s.t.

$$\begin{cases} (u(x,t) - u(-x,t)) = a^2 \left(u(x,t) - u(-x,t)\right), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) - u(-x,0) = 0,\ u_t(x,0) - u_t(-x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Clearly, a solution is $u(x,t) - u(-x,t) = 0$. Because the solution to the wave equation is unique, it is the only required solution. Thus, $u(x,t) = u(-x,t)$, implying even u in x.

(2) For Odd Initial Conditions

We also have that if both ϕ and ψ are odd functions, then so is u in x.

Odd $\phi(x)$ and $\psi(x)$ give that $\phi(x) = -\phi(-x)$ and $\psi(x) = -\psi(-x)$. The associated initial value problem for $u(-x,t)$ is

$$\begin{cases} u_{tt}(-x,t) - a^2 u_{xx}(-x,t) = 0, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(-x,0) = \phi(-x) = -\phi(x),\ u_t(-x,0) = \psi(-x) = -\psi(x), & x \in \mathbb{R}, \end{cases}$$

and consider $u(x,t) + u(-x,t)$ as a linear combination of two solution of the wave equation s.t.

$$\begin{cases} (u(x,t) + u(-x,t)) = a^2 \left(u(x,t) + u(-x,t)\right), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) + u(-x,0) = 0,\ u_t(x,0) + u_t(-x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Clearly, a solution is $u(x,t) + u(-x,t) = 0$. Because the solution to the wave equation is unique, it is the only required solution. Thus, $u(x,t) = -u(-x,t)$, implying odd u in x.

7 (Principle of causality) Let u be a smooth solution of the wave equation $u_{tt} = u_{xx},\ x \in \mathbb{R},\ t \geq 0$. Prove that...

Let u be a smooth solution of the wave equation

$$u_{tt} = u_{xx}, \quad x \in \mathbb{R},\ t \geq 0.$$

Prove that for any $(x_0, t_0) \in (-\infty, +\infty) \times (0, +\infty)$,

$$\int_{x_0-t_0+t}^{x_0+t_0-t} \frac{1}{2}\left(u_t^2 + u_x^2\right)(x,t)\,\mathrm{d}x \leq \int_{x_0-t_0}^{x_0+t_0} \frac{1}{2}\left(u_t^2 + u_x^2\right)(x,0)\,\mathrm{d}x, \quad \forall 0 < t < t_0.$$

Hint: multiply the wave equation by u_t and then try to re-write the equation in the form

$$F_t - G_x = 0.$$

Integrate this equation and apply Green's Theorem to the trapezoid bounded by the x-axis, the characteristic lines passing through (x_0, t_0), and the horizontal line $t = t$.

Proof. Multiply the wave equation by u_t as

$$u_t u_{tt} = u_t u_{xx}$$

and rewrite in the form

$$F_t - G_x = \frac{\partial}{\partial t}\left(\frac{u_t u_t}{2} + \frac{u_x u_x}{2}\right) - \frac{\partial}{\partial x}(u_t u_x)\mathrm{d}x = 0.$$

Integrate this equation we have

$$\int_\Omega F_t - G_x \mathrm{d}A = 0$$

then by Green's Theorem

$$\oint_{\partial\Omega} < F_t, -G_x > \cdot\mathbf{n}\,\mathrm{d}S = 0$$

$$\Rightarrow \int_R < F_t, -G_x > \cdot\mathbf{n}_R \mathrm{d}S + \int_T < F_t, -G_x > \cdot\mathbf{n}_T \mathrm{d}S + \int_M < F_t, -G_x > \cdot\mathbf{n}_M \mathrm{d}S = 0$$

and so

$$\int_T \frac{1}{2}\left(u_t^2 + u_x^2\right)\mathrm{d}S + \int_M < F_t, -G_x > \cdot\mathbf{n}_M \mathrm{d}S = \int_R \frac{1}{2}\left(u_t^2 + u_x^2\right)\mathrm{d}S$$

where Ω is the area bounded by the x-axis (denoted as R), the characteristic lines passing through (x_0, t_0) (denoted as M), and the horizontal line $t = t$ (denoted as T). Let $\xi = x - x_0$, then it is not hard to deduce

$$\int_M < F_t, -G_x > \cdot\mathbf{n}_M \mathrm{d}S = \frac{1}{\sqrt{2}}\int_M \left(\frac{(u_t - u_\xi)^2}{2} + \frac{(u_x - u_\xi)^2}{2}\right)\mathrm{d}S \geq 0.$$

Hence we end up with the inequality

$$\int_{x_0-t_0+t}^{x_0+t_0-t} \frac{1}{2}\left(u_t^2 + u_x^2\right)(x,t)\mathrm{d}x \leq \int_{x_0-t_0}^{x_0+t_0} \frac{1}{2}\left(u_t^2 + u_x^2\right)(x,0)\mathrm{d}x, \quad \forall\, 0 < t < t_0.$$

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