PDE Introductory Exercises and Solutions
Chapter 4, Elliptic Equations

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Contents

1 Derive the free Green's function $G_0(\mathrm{x})$ in 3-D.

Solution. We have the free Green's function of Laplace equation

$$-\Delta G_0(\mathbf{x}) = \delta(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{$*$}$$

In 3-D case, note that $\delta(\mathbf{x})$ is spherically symmetric about the origin. It is reasonable to seek spherically symmetric solution:

$$G_0(\mathbf{x}) = F(r), \quad r = |\mathbf{x}|.$$

With Laplace operator in polar and spherical coordinates, we have

$$G_0(\mathbf{x}) = \frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}F}{\mathrm{d}r}\right).$$

Equation ($*$) leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}F}{\mathrm{d}r}\right) = -r^2\delta(\mathbf{x}),$$

$$\begin{aligned}
r^2\frac{\mathrm{d}F}{\mathrm{d}r} &= -\int_0^r r^2\delta(\mathbf{x})\mathrm{d}r \\
&= -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^r \delta(\mathbf{x})r^2\sin\theta\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \\
&= -\frac{1}{4\pi}\iiint_{|\mathbf{x}|<r}\delta(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= -\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\delta(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= -\frac{1}{4\pi}
\end{aligned}$$

where Ω is the spherical shell and the divergence theorem is applied.

Now we have the ODE $\frac{\mathrm{d}F}{\mathrm{d}r} = -\frac{1}{4\pi r^2}$ and its solution $F(r) = \frac{1}{4\pi r} + C$, where C is a constant. Since constant C is a trivial solution of Laplace equation $\Delta u = 0$, simply take $C = 0$. Finally,

$$G_0(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}.$$

2 A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution u ...

A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution u. Its inner boundary is held at $100°C$. Its outer boundary satisfies

$$\frac{\partial u}{\partial \mathbf{n}} = -\gamma,$$

where γ is a constant.

(1) Find the temperature u. Hint: everything is radial and hence so is u.

The steady-state temperature u follows the Laplace equation. With the Laplacian operator in spherical coordinates, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \psi^2} = 0.$$

Notice that the everything is radial and hence so is u. Then, the PDE simplifies to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = 0.$$

Since it is a first-order ODE for $\mathrm{d}u/\mathrm{d}r$, rewrite the ODE as $\dfrac{\mathrm{d}D}{\mathrm{d}r} + \dfrac{2}{r}D = 0$ where $D = \mathrm{d}u/\mathrm{d}r$. The solution is $D = \dfrac{C_1}{r^2}$ where C_1 is a constant. To determine C_1, use the boundary condition at the outer radius

$$D|_{r=2} = \frac{C_1}{4} = -\gamma$$

thus $C_1 = -4\gamma$. Now we have $\dfrac{\mathrm{d}u}{\mathrm{d}r} = -\dfrac{4\gamma}{r^2}$. The solution is $u(r) = \dfrac{4\gamma}{r} + C_2$ where C_2 is a constant. To determine C_2, use the boundary condition at the inner radius

$$u(1) = 4\gamma + C_2 = 100$$

thus $C_2 = 100 - 4\gamma$. Finally,

$$u(r) = \frac{4\gamma}{r} + 100 - 4\gamma, \quad 1 < r < 2.$$

(2) What are the hottest and coldest temperatures?

By the strong maximum (minimum) value theorem, the maximum and minimum temperatures occur at $r = 1$, $r = 2$. That is, if $\gamma \geq 0$, the hottest temperature takes $u(1) = 100$ and the coldest temperature takes $u(2) = 100 - 2\gamma$. And if $\gamma < 0$, the hottest temperature takes $u(2) = 100 - 2\gamma$ and the coldest temperature takes $u(1) = 100$.

(3) Can you choose γ so that the temperature on the outer boundary is $20°C$?

There is $u(2) = 100 - 2\gamma = 20 \ \Rightarrow \gamma = 40$.

3 Suppose that u is a harmonic function in disk $D = \{r < 2\}$ and that $u = 3\sin(2\theta) + 1$ for $r = 2$ ...

Suppose that u is a harmonic function in disk $D = \{r < 2\}$ and that $u = 3\sin(2\theta) + 1$ for $r = 2$. Without finding the solution, answer the following questions:

(1) Find the maximum value of u in $\overline{D}$;

For $u = 3\sin(2\theta) + 1, \theta \in [0, 2\pi)$, by the maximum principle, it is sufficient to consider u on ∂D:

$$\max_{(r,\theta)\in\partial D} u = 4$$

when $r = 2, \theta = \pi/4$.

(2) Calculate the value of u at the origin.

By the mean value property

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} (3\sin(2\theta) + 1)\,\mathrm{d}\theta = 1.$$

4 Find the Green's function $G(M; M_0)$ for Dirichlet problem in the first quadrant of plane.

$$\begin{cases} -\Delta G = \delta(M - M_0) & \text{in } \Omega = \{M = (x,y); x > 0, y > 0\}, \\ G|_{x=0} = 0, & G|_{y=0} = 0. \end{cases}$$

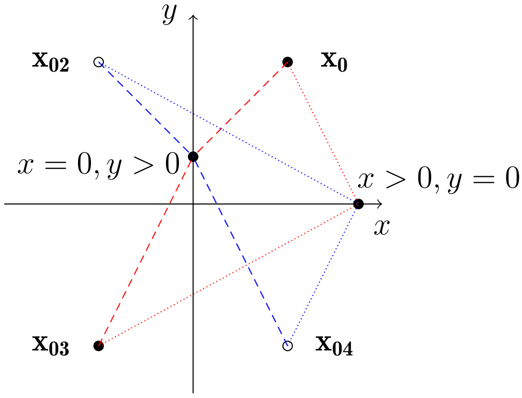
Solution. We have the Green's function for the whole plane

$$G_0(\mathbf{x}) = -\frac{1}{2\pi}\ln|\mathbf{x}|.$$

and so

$$G_0(\mathbf{x} - \mathbf{x_0}) = -\frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x_0}|.$$

Applying the method of images, shown in the figure,



since $G(\mathbf{x}; \mathbf{x_0})$ is identically equal to zero on the positive x-axis and y-axis, we have

$$G(\mathbf{x}; \mathbf{x_0}) = G_0(\mathbf{x} - \mathbf{x_0}) - G_0(\mathbf{x} - \mathbf{x_{02}}) - G_0(\mathbf{x} - \mathbf{x_{04}}) + G_0(\mathbf{x} - \mathbf{x_{03}})$$

where let $\mathbf{x} = (x,y)$ and $\mathbf{x_0} = (x_0, y_0)$ s.t. $\mathbf{x_{02}} = (-x_0, y_0)$, $\mathbf{x_{03}} = (-x_0, -y_0)$ and $\mathbf{x_{04}} = (x_0, -y_0)$. Thus,

$$\begin{aligned} G(\mathbf{x}; \mathbf{x_0}) &= -\frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x_0}| + \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x_{02}}| + \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x_{04}}| - \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x_{03}}| \\ &= -\frac{1}{4\pi}\ln\frac{((x-x_0)^2 + (y-y_0)^2)((x+x_0)^2 + (y+y_0)^2)}{((x+x_0)^2 + (y-y_0)^2)((x-x_0)^2 + (y+y_0)^2)}. \end{aligned}$$

5 Select a suitable method to solve the following boundary value problems.

(1)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (r,\theta); 0 \le r < R, 0 \le \theta < 2\pi\}, \\ u(R,\theta) = A\cos\theta. \end{cases}$$

Solution. With the separation of variables, let the solution $u(x, y) = R(r)\Theta(\theta)$. By the Laplace operator in polar coordinates,

$$\frac{r^2 R'' + rR'}{-R} = \frac{\Theta''}{\Theta} \equiv -\lambda$$

so we have

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0,$$
$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0.$$

The function $\Theta(\theta)$ must be 2π-periodic, as the boundary condition. The eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2, \quad n = 0, 1, 2, \cdots,$$
$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n = 0, 1, 2, \cdots.$$

By solving the Euler's equation, we have

$$R_0(r) = C_0 + D_0 \ln r, \quad R_n(r) = C_n r^n + D_n r^{-n},\ n = 1, 2, \cdots.$$

That is, the solution can be written in the form of

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

The solution $u(r, \theta)$ must be continuous, and then bounded, at $r = 0$. Therefore the coefficients D_n including D_0 must be zero. Then

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} C_n r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

or rewrite the coefficients as

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

By the B.C. $u(R, \theta) = A\cos\theta$, i.e. $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} R^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = A\cos\theta$

$$a_n = 0, \quad n \ne 1,$$
$$a_n = \frac{A}{R}, \quad n = 1,$$
$$b_n = 0, \quad n = 1, 2, \cdots.$$

Hence, the solution is

$$u(r,\theta) = \frac{A}{R} r\cos\theta.$$

(2)

$$\begin{cases} \Delta u = 1 & \text{in } \Omega = \{M = (r, \theta); 0 \le r < R, 0 \le \theta < 2\pi\}, \\ u(R, \theta) = 0. \end{cases}$$

Hint: the region and B.C. are radially symmetric and hence the solution should be radially symmetric.

Solution. Since the region and B.C. are radially symmetric, the solution should be radially symmetric. That is, let $u(r, \theta) = R(r)$ then we have

$$R''(r) + \frac{1}{r} R'(r) = 1.$$

The solution is $R(r) = c_1 \ln(r) + c_2 + \frac{r^2}{4}$. At $r = 0$, the continuous and bounded u gives $c_1 = 0$. By the B.C. $u(R, \theta) = 0$,

$$c_2 + \frac{R^2}{4} = 0 \quad \Rightarrow c_2 = -\frac{R^2}{4}.$$

Thus

$$u(r, \theta) = \frac{r^2}{4} - \frac{R^2}{4}.$$

(3)

$$\begin{cases} \Delta u = \frac{A}{2} r^2 \sin(2\theta) & \text{in } \Omega = \{M = (r, \theta); 0 \le r < R, 0 \le \theta < 2\pi\}, \\ u(R, \theta) = 0. \end{cases}$$

Hint: For each fixed r, $u(r, \theta)$ is $2\pi-$periodic function of θ which can be expanded by the eigenfunctions with 2π-period B.C.. Thus u takes the form of

$$u(r, \theta) = A_0(r) + \sum_{n=1}^{\infty} \left(A_n(r) \cos(n\theta) + B_n(r) \sin(n\theta) \right).$$

Solution. For each fixed r and with thinking of u as a 2π-periodic function of θ, we can write $u(r, \theta)$ as a series involving $A_n \cos(n\theta) + B_n \sin(n\theta), n = 0, 1, 2, \cdots$,

$$u(r, \theta) = A_0(r) + \sum_{n=1}^{\infty} \left(A_n(r) \cos(n\theta) + B_n(r) \sin(n\theta) \right).$$

Apply the Laplace operator in polar coordinates

$$\Delta u = A_0''(r) + \frac{A_0'(r)}{r} + \sum_{n=1}^{\infty} \left(\left(A_n''(r) + \frac{A_n'(r)}{r} - \frac{n^2 A_n(r)}{r^2} \right) \cos(n\theta) \right.$$
$$\left. + \left(B_n''(r) + \frac{B_n'(r)}{r} - \frac{n^2 B_n(r)}{r^2} \right) \sin(n\theta) \right) = \frac{A}{2} r^2 \sin(2\theta).$$

Comparing both sides, we have that all the $A_n = 0, B_n = 0$, except A_0 and B_2 which must satisfy

$$A_0''(r) + \frac{A_0'(r)}{r} = 0,$$
$$B_2''(r) + \frac{B_2'(r)}{r} - \frac{4B_2(r)}{r^2} = \frac{A}{2}r^2.$$

We are familiar with the solution of the A_0-equation as

$$A_0(r) = C + D\ln r.$$

For the B_2-equation, the general solution for the corresponding homogeneous equation

$$B_2''(r) + \frac{B_2'(r)}{r} - \frac{4B_2(r)}{r^2} = 0$$

is given by $c_1r^2 + c_2r^{-2}$ then with a particular solution $\dfrac{Ax^4}{24}$, we have

$$B_2(r) = c_1r^2 + c_2r^{-2} + \frac{Ar^4}{24}.$$

u must be in the form

$$u(r,\theta) = C + D\ln r + \left(c_1r^2 + c_2r^{-2} + \frac{Ar^4}{24}\right)\sin(2\theta).$$

At $r = 0$, the continuous and bounded u promises $D = c_2 = 0$. By the B.C. $u(R,\theta) = 0$,

$$C + \left(c_1R^2 + \frac{AR^4}{24}\right)\sin(0) = 0 \quad \Rightarrow C = 0,$$

and

$$C + \left(c_1R^2 + \frac{AR^4}{24}\right)\sin(2\theta) = 0 \quad \Rightarrow C = -\left(c_1R^2 + \frac{AR^4}{24}\right)\sin(2\theta) = 0$$
$$\Rightarrow c_1R^2 + \frac{AR^4}{24} = 0 \quad \Rightarrow c_1 = -\frac{AR^2}{24}.$$

Finally,

$$u(r,\theta) = \left(-\frac{AR^2r^2}{24} + \frac{Ar^4}{24}\right)\sin(2\theta).$$

(4)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x,y); 0 < x < \pi, 0 < y < \pi\}, \\ u(0,y) = 0, & u(\pi,y) = \cos^2 y, \\ u_y(x,0) = 0, & u_y(x,\pi) = 0. \end{cases}$$

Solution. With the separation of variables, let the solution $u(x,y) = X(x)Y(y)$. By the PDE,

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

and so

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} \equiv -\lambda.$$

We have $Y''(y) + \lambda Y(y) = 0$ and Y satisfies the Neumann B.C. $Y'(0) = Y'(\pi) = 0$. Thus, the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2, \quad n = 0, 1, 2, \cdots,$$

$$Y_n(y) = \cos(ny), \quad n = 0, 1, 2, \cdots.$$

Besides, the solution of $X''(x) - \lambda X(x) = 0$ is $X_n(x) = c_1 e^{nx} + c_2 e^{-nx}, n = 1, 2, \cdots$ and $X_n(x) = c_1 x + c_2$ for $n = 0$. By the B.C. $X(0) = 0$, there must be $c_2 = 0$ for $n = 0$ and $c_2 = -c_1$ for $n = 1, 2, \cdots$, i.e.

$$X_0(x) = a_0 x, \quad X_n(x) = a_n(e^{nx} - e^{-nx}).$$

Now we form

$$u(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y) = a_0 x + \sum_{n=1}^{\infty} a_n(e^{nx} - e^{-nx}) \cos(ny).$$

To satisfy the B.C. $u(\pi, y) = \cos^2 y$, it requires

$$a_0 \pi + \sum_{n=1}^{\infty} a_n(e^{n\pi} - e^{-n\pi}) \cos(ny) = \cos^2 y,$$

from which we have

$$\int_0^{\pi} a_0 \pi \mathrm{d}y = \int_0^{\pi} \cos^2 y \mathrm{d}y = \int_0^{\pi} \frac{1 + \cos(2y)}{2} \mathrm{d}y = \frac{\pi}{2} \quad \Rightarrow a_0 = \frac{1}{2\pi},$$

$$\int_0^{\pi} a_n(e^{n\pi} - e^{-n\pi}) \cos^2(ny) \mathrm{d}y = \int_0^{\pi} \cos^2 y \cos(ny) \mathrm{d}y = \int_0^{\pi} \frac{1 + \cos(2y)}{2} \cos(ny) \mathrm{d}y$$

$$= \begin{cases} \pi/4 & , n = 2, \\ 0 & , n \neq 2. \end{cases} \quad \Rightarrow a_n = \begin{cases} \dfrac{1}{2(e^{2\pi} - e^{-2\pi})} & , n = 2, \\ 0 & , n \neq 2. \end{cases}$$

Hence,

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} x + \frac{(e^{2\pi} - e^{-2\pi})}{2(e^{2\pi} - e^{-2\pi})} \cos(2y) \\ &= \frac{1}{2\pi} x + \frac{1}{2} \cos(2y). \end{aligned}$$

(5)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x, y); 0 < x < a, 0 < y < b\}, \\ u(0, y) = 0, & u(a, y) = 0, \\ \left.\left(\dfrac{\partial u}{\partial y} + u\right)\right|_{y=0} = 0, & u(x, b) = g(x). \end{cases}$$

Solution. With the separation of variables, let the solution $u(x,y) = X(x)Y(y)$. By the PDE,

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

and so

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \equiv -\lambda.$$

We have $X''(x) + \lambda X(x) = 0$ and X satisfies the Dirichlet B.C. $X(0) = X(a) = 0$. Thus, the eigenvalues and eigenfunctions are given by

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, \cdots,$$

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \cdots.$$

Besides, a solution of $Y''(y) - \lambda Y(y) = 0$ is

$$Y_n(y) = c_n \cosh\frac{n\pi y}{a} + d_n \sinh\frac{n\pi y}{a}, \quad n = 1, 2, \cdots.$$

Now we form

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left(c_n \cosh\frac{n\pi y}{a} + d_n \sinh\frac{n\pi y}{a}\right)$$

where the coefficients a_n are given by

$$a_n Y_n(b) = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}\mathrm{d}x.$$

By the boundary condition

$$\begin{aligned}\left.\left(\frac{\partial u}{\partial y} + u\right)\right|_{y=0} &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left(c_n\frac{n\pi}{a}\cdot 0 + d_n\frac{n\pi}{a}\cdot 1\right) + \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)(c_n\cdot 1 + d_n \cdot 0)\\ &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\left(d_n\frac{n\pi}{a} + c_n\right)\\ &= 0\end{aligned}$$

the solution requires

$$d_n\frac{n\pi}{a} + c_n = 0.$$

Also,

$$u(x,b) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\left(c_n\cosh\frac{n\pi b}{a} + d_n\sinh\frac{n\pi b}{a}\right) = g(x)$$

from which we derive

$$\frac{a}{2}\left(c_n\cosh\frac{n\pi b}{a} + d_n\sinh\frac{n\pi b}{a}\right) = \int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.$$

Finally, the coefficients can be solved explicitly as

$$\begin{cases} c_n = \dfrac{2}{a\left(\cosh\dfrac{n\pi b}{a} - \dfrac{a}{n\pi}\sinh\dfrac{n\pi b}{a}\right)}\displaystyle\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x,\\ d_n = \dfrac{2}{a\left(\sinh\dfrac{n\pi b}{a} - \dfrac{n\pi}{a}\cosh\dfrac{n\pi b}{a}\right)}\displaystyle\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.\end{cases}$$

6 Find the solutions that depend only on r of the Helmholtz equation $-\Delta u = \lambda^2 u$ in 3-D, where $\lambda > 0$ is a constant.

Solution. With the Laplacian operator in spherical coordinates, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \psi^2} = -\lambda^2 u.$$

To find solutions that depend only on the radial coordinate r, i.e. solution is spherically symmetrical, it is sufficient to solve an ODE

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = -\lambda^2 u \quad \Rightarrow \frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda^2 u = 0. \tag{$*$}$$

Introduce a new function

$$u(r) = \frac{v(r)}{r}$$

s.t the original equation ($*$) becomes

$$\left(\frac{1}{r}\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} - \frac{2}{r^2}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2v}{r^3}\right) + \frac{2}{r}\left(\frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r^2}\right) + \lambda^2\left(\frac{v}{r}\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{\lambda^2}{r} v = 0 \quad \Rightarrow \frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \lambda^2 v = 0.$$

The general solution to this simplified second-order linear ODE is

$$v(r) = A\sin(\lambda r) + B\cos(\lambda r)$$

where A, B are constants to be determined. Returning to the original u, the solution is given by

$$u(r) = \frac{A\sin(\lambda r)}{r} + \frac{B\cos(\lambda r)}{r}.$$

7 Show that there is no solution of...

Show that there is no solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \subset \mathbb{R}^3, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

unless

$$\int_\Omega f \mathrm{d}x = \int_{\partial\Omega} g \mathrm{d}S.$$

Proof. Suppose that

$$\int_\Omega f \mathrm{d}\mathbf{x} \neq \oint_{\partial\Omega} g \mathrm{d}S. \tag{$*$}$$

By the definition we have

$$\int_\Omega f \mathrm{d}\mathbf{x} = \int_\Omega \Delta u \mathrm{d}\mathbf{x}$$

and

$$\oint_{\partial\Omega} g \mathrm{d}S = \oint_{\partial\Omega} \frac{\partial u}{\partial n} \mathrm{d}S = \oint_{\partial\Omega} \nabla u \cdot \mathbf{n} \mathrm{d}S.$$

However, we see that ($*$) deduces the solution u satisfying

$$\int_\Omega \Delta u \mathrm{d}\mathbf{x} \neq \oint_{\partial\Omega} \nabla u \cdot \mathbf{n} \mathrm{d}S$$

which is contradictory to the Divergence Theorem. ■

8 Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. . .

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider Poisson equation with Neumann boundary condition

$$\begin{cases} -\Delta G_N(\mathbf{x};\mathbf{x_0}) = \delta(\mathbf{x}-\mathbf{x_0}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial G_N(\mathbf{x};\mathbf{x_0})}{\partial \mathbf{n}} = const. \quad C, & \mathbf{x} \in \partial\Omega. \end{cases}$$

(1) Find the value of const. C such that the above boundary value problem has a solution.

Solution. One must have

$$\int_{\partial\Omega} \frac{\partial G_N(\mathbf{x};\mathbf{x_0})}{\partial \mathbf{n}} \mathrm{d}S = \int_\Omega \Delta G_N(\mathbf{x};\mathbf{x_0}) \mathrm{d}V = \int -\delta(\mathbf{x}-\mathbf{x_0}) \mathrm{d}V = -1.$$

Thus,

$$\int_{\partial\Omega} C \mathrm{d}S = -1 \quad \Rightarrow C = -\frac{1}{\displaystyle\int_{\partial\Omega} \mathrm{d}S} \quad \Rightarrow C = -\frac{1}{|\partial\Omega|}.$$

(2) By using G_N, find a formula for $u(x_0)$

where u is a solution of

$$\begin{cases} -\Delta u = f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = g(x), & x \in \partial\Omega. \end{cases}$$

Solution. By Green's second identity,

$$\begin{aligned} &\int_\Omega \left(u(\mathbf{x}) \Delta G_N(\mathbf{x};\mathbf{x_0}) - G_N(\mathbf{x};\mathbf{x_0}) \Delta u(\mathbf{x})\right) \mathrm{d}V \\ = &\int_{\partial\Omega} \left(u(\mathbf{x}) \frac{\partial G_N(\mathbf{x};\mathbf{x_0})}{\partial \mathbf{n}} - G_N(\mathbf{x};\mathbf{x_0}) \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} \right) \mathrm{d}S. \end{aligned}$$

then, after substituting,

$$\int_{\Omega}(-u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x_0})+G_N(\mathbf{x};\mathbf{x_0})f(\mathbf{x}))\,\mathrm{d}V=\int_{\partial\Omega}(u(\mathbf{x})\cdot C-G_N(\mathbf{x};\mathbf{x_0})g(\mathbf{x}))\,\mathrm{d}S.$$

Therefore,

$$-u(\mathbf{x_0})+\int_{\Omega}G_N(\mathbf{x};\mathbf{x_0})f(\mathbf{x})\mathrm{d}V=C\int_{\partial\Omega}u(\mathbf{x})\mathrm{d}S-\int_{\partial\Omega}G_N(\mathbf{x};\mathbf{x_0})g(\mathbf{x})\mathrm{d}S$$

$$u(\mathbf{x_0})=-C\int_{\partial\Omega}u(\mathbf{x})\mathrm{d}S+\int_{\partial\Omega}G_N(\mathbf{x};\mathbf{x_0})g(\mathbf{x})\mathrm{d}S+\int_{\Omega}G_N(\mathbf{x};\mathbf{x_0})f(\mathbf{x})\mathrm{d}V$$

where

$$-C\int_{\partial\Omega}u(\mathbf{x})\mathrm{d}S=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}u(\mathbf{x})\mathrm{d}S$$

is a constant.

9 Consider Poisson equation...

Consider Poisson equation

$$\begin{cases}-\Delta u=f(\mathbf{x}), & x\in\mathbb{R}^3,\\ \lim\limits_{|\mathbf{x}|\to\infty}u=0,\end{cases}$$

where

$$f(x)=\begin{cases}1, & \text{if } |x|\le 1,\\ 0, & \text{if } |x|>1.\end{cases}$$

(1) Solve this equation (leave your answer as an integral).

Since $\lim\limits_{|\mathbf{x}|\to\infty}u=0$ promises the integrability, the solution is given by

$$u(\mathbf{x})=\int_{\mathbb{R}^3}f(\mathbf{x_0})G(\mathbf{x};\mathbf{x_0})\mathrm{d}V_{\mathbf{x_0}}=\int_{\mathbb{R}^3}\frac{1}{4\pi|\mathbf{x}-\mathbf{x_0}|}f(\mathbf{x_0})\mathrm{d}V_{\mathbf{x_0}}.$$

Substituting $f(\mathbf{x})$, it leads to

$$u(\mathbf{x})=\int_{|\mathbf{x_0}|\le 1,\mathbf{x_0}\in\mathbb{R}^3}\frac{1}{4\pi|\mathbf{x}-\mathbf{x_0}|}\mathrm{d}V_{\mathbf{x_0}}.$$

(2) Find $\lim_{|\mathbf{x}|\to\infty}|\mathbf{x}|u(\mathbf{x})$.

$$\lim_{|\mathbf{x}|\to\infty}|\mathbf{x}|u(\mathbf{x})=\lim_{|\mathbf{x}|\to\infty}|\mathbf{x}|\int_{|\mathbf{x_0}|\le 1,\mathbf{x_0}\in\mathbb{R}^3}\frac{1}{4\pi|\mathbf{x}-\mathbf{x_0}|}\mathrm{d}V_{\mathbf{x_0}}=\lim_{|\mathbf{x}|\to\infty}|\mathbf{x}|\cdot\frac{1}{4\pi|\mathbf{x}|}\cdot\frac{4}{3}\pi=\frac{1}{3}.$$

(3) Let c be the limit found in (ii). Then $u(x) \approx \frac{c}{|x|}$ for $|x|$ large. Interpret this physically.

$$u(\mathbf{x}) \approx \frac{1}{3|\mathbf{x}|} \text{ for } |\mathbf{x}| \text{ large.}$$

Physically, this means that when $|\mathbf{x}|$ is larger enough, u satisfying the PDE decays as the reciprocal function ($u(\mathbf{x}) \sim 1/|\mathbf{x}|$). In particular, for the electrostatic field problem, given a uniform spherical charge, such a solution u describes how the potential diminishes with distance when the location is far away enough.

10 (Harnack Inequality) Let u be a nonnegative harmonic function in $\mathbb{R}^n$. Prove that...

Let u be a nonnegative harmonic function in $\mathbb{R}^n$. Prove that

$$\sup_{\mathbb{R}^n} u \leq 2^n \inf_{\mathbb{R}^n} u.$$

Hint: take an arbitrary pair of points P and Q. Let $R = |P - Q|$. Use the mean value property of harmonic functions on the balls $B_R(P)$ and $B_{2R}(Q)$ (balls centered at P and Q with radius R and $2R$, respectively).

Proof. Consider an arbitrary pair of points P and Q and let $R = |P - Q|$.

First, set point P as the center of a sphere with radius R s.t. Q is a point at the surface S of sphere. Applying the Poisson's formula for the n-dimension sphere,

$$u(\mathbf{x}) = \int_S \phi(\xi) \frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R |\mathbf{x} - \xi|^n} \mathrm{d}S_\xi$$

where $\mathbf{x}$ is a point in the sphere and ω_{n-1} is the surface area of the unit $(n-1)$-dimension sphere. Then, with the point $\mathbf{y}$ on the surface, we have

$$u(\mathbf{x}) = \int_S u(\mathbf{y}) \frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R |\mathbf{x} - \mathbf{y}|^n} \mathrm{d}S_{\mathbf{y}}.$$

For the Poisson kernel, by $\max(|\mathbf{x} - \mathbf{y}|) = R + |\mathbf{x}|$ and $\min(|\mathbf{x} - \mathbf{y}|) = R - |\mathbf{x}|$,

$$\frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R (R + |\mathbf{x}|)^n} \leq \frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R |\mathbf{x} - \mathbf{y}|^n} \leq \frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R (R - |\mathbf{x}|)^n}.$$

That is,

$$\frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R (R + |\mathbf{x}|)^n} \int_S u(\mathbf{y}) \mathrm{d}S_{\mathbf{y}} \leq u(\mathbf{x}) \leq \frac{R^2 - |\mathbf{x}|^2}{\omega_{n-1} R (R - |\mathbf{x}|)^n} \int_S u(\mathbf{y}) \mathrm{d}S_{\mathbf{y}}.$$

Use the mean value property of harmonic functions on the ball,

$$u(\mathbf{0}) = \frac{1}{\Omega_{n-1}(R)} \int_S u(\mathbf{y}) \mathrm{d}S_{\mathbf{y}} = \frac{1}{\omega_{n-1} R^{n-1}} \int_S u(\mathbf{y}) \mathrm{d}S_{\mathbf{y}},$$

where $\Omega_{n-1}(R)$ is the surface area of $(n-1)$-dimension sphere with radius R, we have

$$R^{n-2}\frac{R^2-|\mathbf{x}|^2}{(R+|\mathbf{x}|)^n}u(\mathbf{0}) \leq u(\mathbf{x}) \leq R^{n-2}\frac{R^2-|\mathbf{x}|^2}{(R-|\mathbf{x}|)^n}u(\mathbf{0})$$

rewritten as

$$R^{n-2}\frac{R^2-|\mathbf{x_p}|^2}{(R+|\mathbf{x_p}|)^n}u(\mathbf{p}) \leq u(\mathbf{x_p}) \leq R^{n-2}\frac{R^2-|\mathbf{x_p}|^2}{(R-|\mathbf{x_p}|)^n}u(\mathbf{p}).$$

Then, choose a sphere with radius $2R$ centered at Q s.t.

$$2^{n-2}R^{n-2}\frac{4R^2-|\mathbf{x_p}|^2}{(2R+|\mathbf{x_p}|)^n}u(\mathbf{q}) \leq u(\mathbf{x_p}) \leq 2^{n-2}R^{n-2}\frac{4R^2-|\mathbf{x_p}|^2}{(2R-|\mathbf{x_p}|)^n}u(\mathbf{q}).$$

Thus,

$$R^{n-2}\frac{R^2-|\mathbf{x_p}|^2}{(R+|\mathbf{x_p}|)^n}u(\mathbf{p}) \leq 2^{n-2}R^{n-2}\frac{4R^2-|\mathbf{x_p}|^2}{(2R-|\mathbf{x_p}|)^n}u(\mathbf{q})$$
$$\Rightarrow u(\mathbf{p}) \leq 2^{n-2}u(\mathbf{q})\left(\frac{4R^2-|\mathbf{x_p}|^2}{R^2-|\mathbf{x_p}|^2}\right) / \left(\frac{(2R-|\mathbf{x_p}|)^n}{(R+|\mathbf{x_p}|)^n}\right).$$

For the whole space $\mathbb{R}^n$, let R tend to infinity so that

$$u(\mathbf{p}) \leq 2^{n-2}u(\mathbf{q})\cdot 4 = 2^n u(\mathbf{q}).$$

Since $\sup\limits_{\mathbb{R}^n} u < +\infty$ is implied by above inequality, let $\sup\limits_{\mathbb{R}^n} u = u(\mathbf{p})$ and $\inf\limits_{\mathbb{R}^n} u = u(\mathbf{q})$, then

$$\sup_{\mathbb{R}^n} u \leq 2^n \inf_{\mathbb{R}^n} u.$$

∎

11 (Liouville Theorem) Prove that any harmonic function u in the whole $\mathbb{R}^n$ that is either bounded from below or above must be a constant function...

Prove that any harmonic function u in the whole $\mathbb{R}^n$ that is either bounded from below or above must be a constant function. Hint: Consider either $u - \inf u$ or $\sup u - u$.

Proof. Suppose u is a harmonic function on $\mathbb{R}^n$, bounded by M. Let $x \in \mathbb{R}^n$ and let $r > 0$. By the ball-volume version of the mean-value property

$$|u(\mathbf{x}) - u(\mathbf{0})| = \frac{1}{V(B(\mathbf{0},r))}\left|\int_{B(\mathbf{x},r)} u\mathrm{d}V - \int_{B(\mathbf{0},r)} u\mathrm{d}V\right| \leq M\frac{V(B(\mathbf{0},r)\Delta B(\mathbf{x},r))}{V(B(\mathbf{0},r))}$$

where $B(\mathbf{0},r)\Delta B(\mathbf{x},r)$ means the symmetric difference of the two balls i.e.

$$B(\mathbf{0},r)\Delta B(\mathbf{x},r) = (B(\mathbf{0},r)\cup B(\mathbf{x},r)) \setminus (B(\mathbf{0},r)\cap B(\mathbf{x},r)).$$

Notice $V(B(\mathbf{0},r)\Delta B(\mathbf{x},r)) \to V(B(\mathbf{0},r))$ as $r \to \infty$, thus it forces

$$|u(\mathbf{x}) - u(\mathbf{0})| = 0 \quad \Rightarrow u(\mathbf{x}) = u(\mathbf{0})$$

which says that u is constant.

∎

12 (Decay rate of harmonic functions) Suppose u is harmonic in the exterior of the ball $B_R(0)$ in $\mathbb{R}^3$...

Suppose u is harmonic in the exterior of the ball $B_R(0)$ in $\mathbb{R}^3$ such that it decays at infinity:

$$\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0.$$

(1) Define $v(\mathbf{x}) = MG_0(\mathbf{x}) - u(\mathbf{x})$ where G_0 is the fundamental solution of Laplace equation...

$$v(\mathbf{x}) = MG_0(\mathbf{x}) - u(\mathbf{x}),$$

where G_0 is the fundamental solution of Laplace equation, and the constant M is taken large enough such that $v > 0$ on $\partial B_R(0)$. Prove that v is positive in the exterior of $B_R(0)$. Hint: argue by contradiction and use the strong minimum principle.

Proof. Suppose $v \le 0$ in the exterior of $B(\mathbf{0}, R)$, we have

$$-\Delta v = -M\Delta G_0(\mathbf{x}) + \Delta u(\mathbf{x}) \ge 0.$$

In the region $\mathbb{R}^3 \backslash (B(\mathbf{0}, R) \backslash \partial B(\mathbf{0}, R))$, the strong minimum principle promise that the minimum function value of v is taken only on the boundary $\partial B(\mathbf{0}, R)$, since $\lim\limits_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$ gives that v is not a constant function here.

That is, in the exterior of $B(\mathbf{0}, R)$, $v > \min\limits_{\mathbf{x}\in\partial B(\mathbf{0},R)} v(\mathbf{x}) > 0$, which means that the assumption $v \le 0$ is impossible.

■

(2) Prove that u decays at infinity at least as fast as the fundamental solution.

Proof. In 3-D case, there is

$$G_0(\mathbf{x}_0) = \frac{1}{4\pi|\mathbf{x}|} > 0.$$

Assume that $u(\mathbf{x})$ does not decay at least as fast as the fundamental solution at infinite, i.e., for an arbitrary constant $C > 0$, there exists $r > 0$ such that $u(\mathbf{x}) > CG_0(\mathbf{x})$ for all $|\mathbf{x}| > r$ (in the exterior of $B(\mathbf{0}, r)$).

Take $C = M$ and an $\mathbf{x}_0 \in \mathbb{R}^3$ s.t. $|\mathbf{x}_0| > R_0 = \max\{R, r\}$ satisfying above statement, we have

$$v(\mathbf{x}_0) = MG_0(\mathbf{x}_0) - u(\mathbf{x}_0) = CG_0(\mathbf{x}_0) - u(\mathbf{x}_0) < 0$$

which is contradictory to the result of **(1)**.

■