PDE Introductory Exercises and Solutions
Chapter 3, Parabolic Equations

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Contents

1 (Transmission conditions) Consider a surface S that separates two media with different thermal conductivities k_1 and k_2 ...

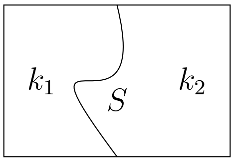
Consider a surface S that separates two media with different thermal conductivities k_1 and k_2. Let u_1 and u_2 be the temperature in the media. Suppose the media are in intimate contact along the surface S so we have

$$u_1 = u_2 \text{ on } S. \tag{1}$$

Prove that on S,

$$-k_1 \frac{\partial u_1}{\partial \mathbf{n}} = -k_2 \frac{\partial u_2}{\partial \mathbf{n}}, \tag{2}$$

where $\mathbf{n}$ is the unit normal vector field of the surface S. ((1) and (2) are called transmission conditions.) Hint: Take an arbitrary patch ΔS of S, and think about the rate at which thermal energy crosses the patch in the direction of the normal.



Proof. Consider a small arbitrary area element ΔS on the surface S. The heat flux across ΔS is given by Fourier's law

$$\mathbf{q}_1 = -k_1 \nabla u_1, \quad \mathbf{q}_2 = -k_2 \nabla u_2,$$

where $\mathbf{q}_1, \mathbf{q}_2$ are the local heat flux densities. Then, the thermal energy across the whole S should be conserved which implies that along the normal of each ΔS

$$F_1 = \mathbf{q}_1 \cdot \mathbf{n} = -k_1 \nabla u_1 \cdot \mathbf{n}, \quad F_2 = \mathbf{q}_2 \cdot \mathbf{n} = -k_2 \nabla u_2 \cdot \mathbf{n}.$$

That is, $F_1 = F_2$ gives $-k_1 \nabla u_1 \cdot \mathbf{n} = -k_2 \nabla u_2 \cdot \mathbf{n}$. Denoted $\mathbf{n} = (n_x, n_y, n_z)$ as the unit normal vector of ΔS, there is

$$-k_1 \nabla u_1 \cdot \mathbf{n} = -k_1 \left(\frac{\partial u_1}{\partial x}, \frac{\partial u_1}{\partial y}, \frac{\partial u_1}{\partial z} \right) \cdot (n_x, n_y, n_z),$$

$$-k_2 \nabla u_2 \cdot \mathbf{n} = -k_2 \left(\frac{\partial u_2}{\partial x}, \frac{\partial u_2}{\partial y}, \frac{\partial u_2}{\partial z} \right) \cdot (n_x, n_y, n_z),$$

which means the desired transmission condition

$$-k_1 \frac{\partial u_1}{\partial \mathbf{n}} = -k_2 \frac{\partial u_2}{\partial \mathbf{n}}.$$

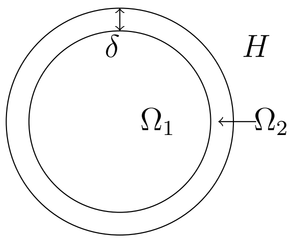
■

2 (Effective boundary condition on a coated body)

Let a body Ω_1 (space shuttle or turbine engine) be thermally insulated by a thin coating Ω_2 of thickness δ; assume the outer boundary of the coating is subject to a high exterior temperature H. Let u_1 be the temperature function in Ω_1 and u_2 be that in Ω_2 that satisfies **1.**(1) on $\partial\Omega_1$. Let the thermal conductivities of the body and the coating be k_1 and k_2, respectively. Prove that on the boundary $\partial\Omega_1$ of the body, we have approximately Robin boundary condition

$$k_1 \frac{\partial u_1}{\partial \mathbf{n}} + \frac{k_2}{\delta}(u_1 - H) = 0, \tag{3}$$

where $\mathbf{n}$ is the unit outer normal vector field of $\partial\Omega_1$. (Equation (3) is called the effective boundary condition; its significance is that with it we do not need to solve, analytically or numerically, the heat equation inside the coating—we just need to solve it inside the body with (3) as the B.C.) To insulate the body well, what should be the scaling relationship of k_2 and δ? Hint: start with **1.**(2); fix a point x on $\partial\Omega_1$, and define $f(\tau) = u_2(x + \tau\mathbf{n})$. Then perform a Taylor expansion of f at 0.



Proof. For a point x on $\partial\Omega_1$, define $f(\tau) = u_2(x + \tau\mathbf{n})$ where τ varies from 0 to δ and f satisfies 1-dimension heat equation $k_2 f''(\tau) = 0$. The Taylor expansion of f at 0 gives that

$$f(\tau) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(\tau - 0)^n = f(0) + f'(0)\tau + O(\tau^2) + \cdots$$

and $f(0) = u_2(x) = u_1(x)$, $f'(0) = \nabla u_2(x) \cdot \mathbf{n} = \partial u_2(x)/\partial \mathbf{n}$. After substituting,

$$f(\tau) = u_2(x) + \partial u_2(x)/\partial \mathbf{n} \cdot \tau$$

which implies

$$\frac{\partial u_2}{\partial \mathbf{n}} \approx \frac{H - u_1}{\delta}$$

by the small δ and boundary condition $f(\delta) = H$. Then, on $\partial\Omega_1$ we have the transmission condition

$$-k_1 \frac{\partial u_1}{\partial \mathbf{n}} = -k_2 \frac{\partial u_2}{\partial \mathbf{n}} \quad \Rightarrow k_1 \frac{\partial u_1}{\partial \mathbf{n}} = \frac{k_2(H - u_1)}{\delta}.$$

Thus

$$k_1 \frac{\partial u_1}{\partial \mathbf{n}} + \frac{k_2}{\delta}(u_1 - H) = 0.$$

■

3 Solve the following eigenvalue problems.

(1)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & -l < x < l, \\ X'(-l) = 0, \quad X(l) = 0. \end{cases}$$

Solution. Let $\xi = x + l$ and $X(x) = X(\xi - l) = Y(\xi)$ s.t.

$$\begin{cases} Y''(\xi) + \lambda Y(\xi) = 0, & 0 < \xi < 2l, \\ Y'(0) = 0, Y(2l) = 0. \end{cases}$$

The eigenvalues and eigenfunctions are given by

$$\lambda_n = \left(\frac{(n + \frac{1}{2})\pi}{2l} \right)^2, \quad n = 0, 1, 2, \cdots$$

and

$$Y_n(\xi) = \cos\left(\frac{(n + \frac{1}{2})\pi}{2l} \xi \right), \quad n = 0, 1, 2, \cdots .$$

The solution of the original problem is

$$\lambda_n = \left(\frac{(n + \frac{1}{2})\pi}{2l} \right)^2, \quad n = 0, 1, 2, \cdots$$

and

$$X_n(x) = \cos\left(\frac{(n + \frac{1}{2})\pi}{2l} (x + l) \right), \quad n = 0, 1, 2, \cdots .$$

(2)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(x) \text{ is a periodic function with period } l. \end{cases}$$

Solution. Since $X(x)$ is a periodic function with period l, $X(0) = X(l), X'(0) = X'(l)$. By the general solution of the ODE (maybe in an exponential form with imaginary numbers),

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \quad X'(x) = C_1\sqrt{-\lambda} e^{\sqrt{-\lambda}x} - C_2\sqrt{-\lambda} e^{-\sqrt{-\lambda}x}$$

and the boundary condition,

$$\begin{cases} C_1 + C_2 = C_1 e^{\sqrt{-\lambda}l} + C_2 e^{-\sqrt{-\lambda}l}, \\ C_1\sqrt{-\lambda} - C_2\sqrt{-\lambda} = C_1\sqrt{-\lambda} e^{\sqrt{-\lambda}l} - C_2\sqrt{-\lambda} e^{-\sqrt{-\lambda}l}, \end{cases}$$

that is,

$$\begin{cases} C_1\left(1 - e^{\sqrt{-\lambda}l}\right) + C_2\left(1 - e^{-\sqrt{-\lambda}l}\right) = 0, \\ C_1\left(1 - e^{\sqrt{-\lambda}l}\right) + C_2\left(e^{-\sqrt{-\lambda}l} - 1\right) = 0. \end{cases}$$

For nonzero solution $\{C_1, C_2\}$, one must have

$$\begin{vmatrix} 1-e^{\sqrt{-\lambda}l} & 1-e^{-\sqrt{-\lambda}l} \\ 1-e^{\sqrt{-\lambda}l} & e^{-\sqrt{-\lambda}l}-1 \end{vmatrix} = 0,$$

then $2 = e^{\sqrt{-\lambda}l} + e^{-\sqrt{-\lambda}l}$ i.e. $\cosh\left(\sqrt{-\lambda}l\right) = 1$ which has complex solutions

$$\sqrt{-\lambda}l = 2i\pi k, \quad k \in \mathbb{Z}.$$

This leads to

$$\lambda_k = -\left(\frac{2\pi k}{l} i\right)^2, \quad k \in \mathbb{Z},$$

with $\lambda = 0$ (the equation reduces to $X''(x) = 0$ here), $k = 0, X(x) = a_0 x + b_0$, and $\lambda \neq 0$, $k^2 = (-k)^2$, there is

$$\lambda_n = \left(\frac{2\pi n}{l}\right)^2, \quad n = 0, 1, 2, \cdots.$$

The corresponding eigenfunctions are given by

$$X_n(x) = C_1 e^{2i\pi n x/l} + C_2 e^{-2i\pi n x/l} = a_n \cos\left(\frac{2\pi n x}{l}\right) + b_n \sin\left(\frac{2\pi n x}{l}\right), \quad n = 0, 1, 2, \cdots$$

where a_n, b_n are constants to be determined.

4 Find the eigenvalues of the following problem graphically...

Find the eigenvalues of the following problem graphically

$$\begin{cases} X'' + \lambda X = 0, \quad x \in (0, l), \\ X(0) = 0, \\ X'(l) + hX(l) = 0 \end{cases}$$

where h is a nonzero constant that may not be positive. Note that negative eigenvalues may appear. In this case, what is the behavior of the solution for the corresponding initial-boundary value problem for the homogeneous heat equation?

Solution.
(a) For the case $\lambda > 0$, we have the general solution of the ODE in the eigenvalue problem

$$X(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x).$$

That is, by the boundary condition $X(0) = 0$,

$$X(x) = \sin(\sqrt{\lambda}x).$$

And the boundary condition $X'(l) + hX(l) = 0$ leads to

$$\sqrt{\lambda}\cos(\sqrt{\lambda}l) + h\sin(\sqrt{\lambda}l) = 0,$$

i.e.

$$\frac{\mu}{hl} = -\tan\mu, \quad \text{where } \mu = \sqrt{\lambda} l.$$

Then

$$\begin{cases} \lambda_n = \left(\frac{\mu_n}{l}\right)^2 & n = 1, 2, 3, \cdots, \\ X_n(x) = \sin\frac{\mu_n x}{l} & n = 1, 2, 3, \cdots, \end{cases}$$

are infinitely many eigenvalues and eigenfunctions of this problem.

(b) For the case $\lambda = 0$, if $h \neq -1/l$, $\lambda = 0$ can not be an eigenvalue. Or if $h = -1/l$, $\lambda = 0$ is the eigenvalue and $X(x) = x$ is the eigenfunction.

(c) For the case $\lambda < 0$, the general solution of the ODE in the eigenvalue problem is also given by

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x},$$

with $X'(x) = C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$. The boundary conditions gives

$$\begin{cases} C_1 + C_2 = 0, \\ C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} + h\left(C_1 e^{\sqrt{-\lambda}l} + C_2 e^{-\sqrt{-\lambda}l}\right) = 0, \end{cases}$$

then

$$-C_2\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}l} + e^{-\sqrt{-\lambda}l}\right) + hC_2\left(-e^{\sqrt{-\lambda}l} + e^{-\sqrt{-\lambda}l}\right) = 0.$$

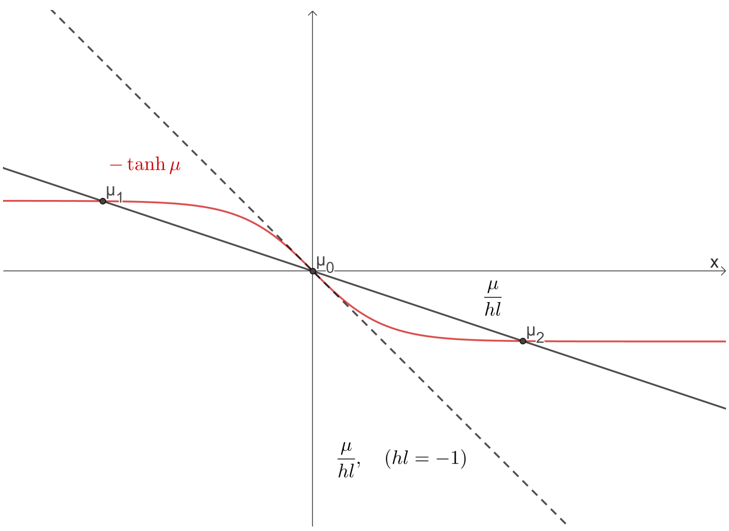
Letting $\sqrt{-\lambda}l = \mu$, there is

$$C_2\sqrt{-\lambda}\cosh\mu + C_2 h\sinh\mu = 0 \quad \Rightarrow \sqrt{-\lambda}\cosh\mu + h\sinh\mu = 0$$

by nonzero C_2. Thus,

$$\frac{\mu}{hl} = -\tanh\mu.$$

This equation has 0 or 2 nonzero real roots as illustrated below, if the roots exist (i.e., at $\mu = 0$, the slope of the straight line $y = \frac{\mu}{hl}$ is larger than the slope of $y = -\tanh(\mu)$) then we denote $\mu_1 = -\mu_2 < 0$.



There is a unique negative eigenvalue $\lambda = -\left(\frac{\mu}{l}\right)^2$ corresponding to the two real roots. Then, the eigenvalues and eigenfunctions are given by

$$\begin{cases} \lambda_n = -\left(\frac{\mu_n}{l}\right)^2, & n = 1, 2, 3, \cdots, \\ X_n(x) = e^{\frac{\mu_n}{l}x} - e^{-\frac{\mu_n}{l}x}, & n = 1, 2, 3, \cdots. \end{cases}$$

In this case, $\lambda_1 < 0 < \lambda_2 < \lambda_3 < \cdots$. And the solution for the corresponding initial-boundary value problem for the homogeneous heat equation will grow exponentially since there is a term $\phi_1 e^{-a^2\lambda_1 t} X_1(x) \to +\infty$ as $t \to +\infty$.

5 Solve the following boundary-initial value problems.

(1)

$$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, \quad t > 0, \\ u(0,t) = u_1, & u(l,t) = u_2, \quad t > 0, \\ u(x,0) = u_0, & 0 < x < l. \end{cases}$$

where u_0, u_1 and u_2 are constants. After solving it, find the limit of $u(x,t)$ as $t \to \infty$. Show that the limit is a steady-state, i.e. a time-independent solution of the PDE and B.C.

Solution. Since the B.C. is non-homogeneous, let $w(x,t) = u_1 \frac{x-l}{0-l} + u_2 \frac{x-0}{l-0} = u_1 + \frac{x}{l}(u_2 - u_1)$ and $u(x,t) = U(x,t) + w(x,t)$. The original problem now is transformed to the problem for $U(x,t)$ with homogeneous B.C.

$$\begin{cases} U_t = a^2 U_{xx}, & 0 < x < l, \quad t > 0, \\ U(0,t) = 0, U(l,t) = 0, & t > 0, \\ U(x,0) = u(x,0) - w(x,0) = u_0 - u_1 + \frac{x}{l}(u_2 - u_1), & 0 < x < l. \end{cases}$$

Let $U(x,t) = X(x) \cdot T(t)$ then $\frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2 T(t)} \equiv -\lambda$. The eigenvalue problem is given by

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < l, \\ X(0) = 0, X(l) = 0. \end{cases}$$

The eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots$$

then the eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \cdots.$$

Expand the solution as

$$U(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$$

and let

$$\Phi(x)=U(x,0)=\sum_{n=1}^{\infty}\Phi_nX_n(x)$$

where

$$\begin{aligned}\Phi_n&=\frac{\displaystyle\int_0^l\Phi(x)X_n(x)\mathrm{d}x}{\displaystyle\int_0^l\left(X_n(x)\right)^2\mathrm{d}x}=\frac{\displaystyle\int_0^l\Big(u_0-u_1+\frac{x}{l}(u_2-u_1)\Big)\sin\frac{n\pi x}{l}\mathrm{d}x}{\dfrac{l}{2}}\\
&=\frac{2}{l}\left((u_0-u_1)\int_0^l\sin\frac{n\pi x}{l}\mathrm{d}x+\frac{u_2-u_1}{l}\int_0^l x\sin\frac{n\pi x}{l}\mathrm{d}x\right)\\
&=\frac{2}{l}\left((u_0-u_1)\frac{l-l\cos(\pi n)}{\pi n}+\frac{u_2-u_1}{l}\frac{l^2(\sin(\pi n)-\pi n\cos(\pi n))}{\pi^2n^2}\right)\\
&=2\left((u_0-u_1)\frac{1-\cos(\pi n)}{\pi n}+(u_2-u_1)\frac{\sin(\pi n)-\pi n\cos(\pi n)}{\pi^2n^2}\right)\quad.\end{aligned}$$

Substituting above into the PDE, we have

$$\sum_{n=1}^{\infty}T_n'(t)X_n(x)=a^2\sum_{n=1}^{\infty}(T_n(t)X_n''(x))=a^2\sum_{n=1}^{\infty}(-\lambda_nT_n(t)X_n(x)).$$

By the B.C.,

$$\sum_{n=1}^{\infty}T_n(0)X_n(x)=\sum_{n=1}^{\infty}\Phi_nX_n(x).$$

Multiply both sides by $X_k(x),k=1,2,\cdots$ and integrate on $[0,l]$, that is,

$$\int_0^l X_k(x)\sum_{n=1}^{\infty}T_n'(t)X_n(x)\mathrm{d}x=\int_0^l X_k(x)a^2\sum_{n=1}^{\infty}(-\lambda_nT_n(t)X_n(x))\mathrm{d}x$$

$$\Rightarrow\sum_{n=1}^{\infty}T_n'(t)\int_0^l X_k(x)X_n(x)\mathrm{d}x=\sum_{n=1}^{\infty}-\lambda_na^2T_n(t)\int_0^l X_k(x)X_n(x)\mathrm{d}x$$

and

$$\int_0^l X_k(x)\sum_{n=1}^{\infty}T_n(0)X_n(x)\mathrm{d}x=\int_0^l X_k(x)\sum_{n=1}^{\infty}\Phi_nX_n(x)\mathrm{d}x$$

$$\Rightarrow\sum_{n=1}^{\infty}T_n(0)\int_0^l X_k(x)X_n(x)\mathrm{d}x=\sum_{n=1}^{\infty}\Phi_n\int_0^l X_k(x)X_n(x)\mathrm{d}x$$

where the order of series summation and integration can be exchanged because we already know that the series is uniformly convergent.

By the orthogonality $\int_a^b X_n(x)X_m(x)\mathrm{d}x=0$, for all $n\neq k$, $\int_0^l X_k(x)X_n(x)\mathrm{d}x=0$ s.t. $\forall\,k$,

$$\begin{cases}T_k'(t)=-a^2\lambda_kT_k(t)\\ T_k(0)=\Phi_k\end{cases}.$$

Solving this initial value problem for the linear ODE, we obtain

$$T_n(t) = \Phi_n e^{-a^2\lambda_n t}, \quad n = 1, 2, \cdots$$

then $U(x,t) =$

$$\sum_{n=1}^{\infty} 2\left((u_0 - u_1)\frac{1-\cos(\pi n)}{\pi n} + (u_2 - u_1)\frac{\sin(\pi n) - \pi n\cos(\pi n)}{\pi^2 n^2}\right) e^{-a^2\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi x}{l}.$$

Thus, $u(x,t) =$

$$\sum_{n=1}^{\infty} 2\left((u_0 - u_1)\frac{1-(-1)^n}{\pi n} + (u_2 - u_1)\frac{(-1)^n}{\pi n}\right) e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi x}{l}$$
$$+ u_1 + \frac{x}{l}(u_2 - u_1).$$

For the limit of $u(x,t)$ as $t \to \infty$, clearly $U(x,t) \to 0$ as $t \to \infty$ which leads to $u(x,t) \to u_1 + \frac{x}{l}(u_2 - u_1)$ as a steady-state.

(2)

$$\begin{cases} u_t = a^2 u_{xx} - hu + g, & 0 < x < l, \quad t > 0, \\ u(0,t) = 0, & u(l,t) = 0, \quad t > 0, \\ u(x,0) = 0, & 0 < x < l. \end{cases}$$

where g and h are constants.

Solution. Let $u(x,t) = X(x) \cdot T(t)$ then $X(x)T'(t) = a^2 X''(x)T(t) - hX(x)T(t) \Rightarrow \frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} - h \Rightarrow \frac{T'(t)}{a^2T(t)} + h = \frac{X''(x)}{X(x)} \equiv -\lambda$ since the PDE is non-homogeneous. The eigenvalue problem is given by

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < l, \\ X(0) = 0, X(l) = 0. \end{cases}.$$

The eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots$$

then the eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \cdots.$$

Expand the solution as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

and let

$$f(x,t) = g = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

where

$$f_n(t)=\frac{\displaystyle\int_0^l f(x,t)X_n(x)\mathrm{d}x}{\displaystyle\int_0^l (X_n(x))^2\,\mathrm{d}x}=\frac{\displaystyle g\int_0^l \sin\frac{n\pi x}{l}\mathrm{d}x}{\dfrac{l}{2}}=\frac{2g(1-\cos(\pi n))}{\pi n}.$$

Substituting above into the PDE, we have

$$\begin{aligned}\sum_{n=1}^{\infty}X_n(x)T_n'(t)&=\sum_{n=1}^{\infty}\left(a^2X_n''(x)T_n(t)-hX_n(x)T_n(t)+f_n(t)\right)\\&=\sum_{n=1}^{\infty}\left((-\lambda a^2-h)X_n(x)T_n(t)+f_n(t)\right).\end{aligned}$$

By the B.C., $\Phi_n=0$ and

$$\sum_{n=1}^{\infty}T_n(0)X_n(x)=0.$$

Similar to **5.(1)**, multiplying both sides of each of the above equation by $X_k(x), k=1,2,\cdots$, integrating on $[0,l]$, and using the orthogonality, we have $\forall\, k$,

$$\begin{cases}T_k'(t)=(-\lambda_k a^2-h)T_k(t)+f_k(t),\\ T_k(0)=0.\end{cases}$$

Solving this initial value problem for the linear ODE, we obtain

$$\begin{aligned}T_n(t)&=\int_0^t e^{(-\lambda_n a^2-h)(t-\tau)}f_n(\tau)\mathrm{d}\tau=\frac{2g(1-\cos(\pi n))}{\pi n}\int_0^t e^{(-\lambda_n a^2-h)(t-\tau)}\mathrm{d}\tau\\&=\frac{2g(1-\cos(\pi n))}{\pi n}\left(-\frac{e^{(-\lambda_n a^2-h)t}}{\lambda_n a^2+h}+\frac{1}{\lambda_n a^2+h}\right).\end{aligned}$$

Then

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}\frac{2g(1-\cos(\pi n))}{\pi n}\left(-\frac{e^{(-\left(\frac{n\pi}{l}\right)^2a^2-h)t}}{\left(\frac{n\pi}{l}\right)^2a^2+h}+\frac{1}{\left(\frac{n\pi}{l}\right)^2a^2+h}\right)\sin\frac{n\pi x}{l}\\&=\sum_{n=1}^{\infty}\frac{2g(1-(-1)^n)}{\pi n}\left(\frac{1-e^{-\left(\left(\frac{n\pi}{l}\right)^2a^2+h\right)t}}{\left(\frac{n\pi}{l}\right)^2a^2+h}\right)\sin\frac{n\pi x}{l}.\end{aligned}$$

(3)

$$\begin{cases}u_t=k^2(u_{xx}+u_{yy}), & 0<x<a,\quad 0<y<b,\quad t>0,\\ u(0,y,t)=u(a,y,t)=0,\\ u(x,0,t)=u(x,b,t)=0,\\ u(x,y,0)=xy.\end{cases}$$

Solution. Start with separation of variables by letting $u(x,y,t)=X(x)Y(y)T(t)$ s.t.

$$\frac{T'(t)}{k^2T(t)}=\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\mu,\quad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}-\mu=\lambda,$$

where λ and μ are constants.

Solve the eigenvalue problems of X, Y and compute the double Fourier coefficients C_{mn}, then solve the ODE about $T_{mn}(t)$. These and the boundary conditions lead to the following eigenvalue

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < a, \\ X(0) = 0, \quad X(a) = 0. \end{cases} \quad \text{and} \quad \begin{cases} Y''(y) + (\mu - \lambda) Y(y) = 0, & 0 < y < b, \\ Y(0) = 0, \quad Y(b) = 0. \end{cases}$$

The eigenvalues and eigenfunctions for (λ_n, X_n) are

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \ldots$$

The eigenvalues and eigenfunctions for $(\mu_{m,n}, Y_m)$ are

$$\mu_{m,n} = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2, \quad Y_m(y) = \sin\left(\frac{m\pi y}{b}\right), \quad m = 1, 2, \ldots$$

Now, write u in a form of series

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n(x) Y_m(y) T_{m,n}(t).$$

Then, by the initial condition we have

$$xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right),$$

where the double Fourier coefficients are determined by

$$c_{m,n} = \frac{4}{l^2}\int_0^l x \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \int_0^l y \sin\left(\frac{m\pi y}{b}\right) \mathrm{d}y = (-1)^{(m+n)} \frac{4ab}{mn\pi^2}.$$

A substitution of these yields the following ODE

$$\begin{cases} T'_{m,n}(t) + \mu_{m,n} k^2 T_{m,n}(t) = 0, & m, n = 1, 2, 3, \cdots, \\ T_{m,n}(0) = c_{m,n}. \end{cases}$$

The solution of the ODE is given by

$$T_{m,n}(t) = c_{m,n} e^{-\mu_{m,n} k^2 t} = (-1)^{(m+n)} \frac{4ab}{mn\pi^2} e^{-\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) k^2 \pi^2 t}.$$

Finally,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (-1)^{(m+n)} \frac{4ab}{mn\pi^2} e^{-\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) k^2 \pi^2 t} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right).$$

6 Consider the initial-Neumann boundary value problem...

Consider the initial-Neumann boundary value problem

$$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, \quad t > 0, \\ u_x(0,t) = 0 = u_x(l,t), & t > 0, \\ u(x,0) = x, & 0 < x < l. \end{cases}$$

Find the limit of $u(x,t)$ as $t \to \infty$ by inspecting the general solution formula obtained by separation of variables. (You do not need to compute all the Fourier coefficients.) Interpret your result physically; generalize it, without proof, to the case of general initial value and higher spatial dimensions.

Solution. Assuming that $u(x,t) = X(x)T(t)$, the initial-Neumann boundary value problem becomes

$$X(x)T'(t) = a^2 X''(x)T(t) \quad \Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(x)}{a^2 T(x)} = -\lambda,$$

then we have two separated equations

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ T'(t) + a^2 \lambda T(t) = 0, & t > 0, \end{cases}$$

and the initial conditions

$$X'(0) = 0 = X'(l).$$

With $\lambda > 0$ required, the solution of ODE $X''(x) + \lambda X(x) = 0, 0 < x < l$ is

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

The boundary conditions gives the eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, 1, 2, \cdots,$$

$$X_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \cdots.$$

Since $T_n(t) = \phi_n \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right)$,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} \phi_n \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \cos\left(\frac{n\pi x}{l}\right).$$

We need to choose the coefficients ϕ_n such that

$$u(x,0) = \sum_{n=1}^{\infty} \phi_n \cos\left(\frac{n\pi x}{l}\right) = x.$$

There is

$$\phi_n = \begin{cases} \dfrac{1}{l}\displaystyle\int_0^l x\mathrm{d}x = \dfrac{l}{2}, & n = 0, \\ \dfrac{2}{l}\displaystyle\int_0^l x\cos\left(\dfrac{n\pi x}{l}\right)\mathrm{d}x = \dfrac{2l\left((-1)^n - 1\right)}{n^2\pi^2}, & n = 1, 2, 3, \cdots. \end{cases}$$

Hence, the solution of the original problem is

$$u(x,t) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l\left((-1)^n - 1\right)}{n^2\pi^2} \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right)\cos\left(\frac{n\pi x}{l}\right).$$

Physically, the B.C. means that there is no temperature difference at each end of the interval $[0, l]$ for all t, i.e., the total thermal energy is conserved in the interval. It implies that temperature will approach a same steady value, that is, the average of the initial temperature. Generally, in a higher spatial dimension, we have

$$\lim_{t\to\infty} u(x,t) = \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x}, 0)\mathrm{d}\mathbf{x}.$$

7 Solve the Cauchy problem for the heat equation... Show that u decays as $t \to \infty$ and find the decay rate. Explain, physically, why u decays as $t \to \infty$.

Solve the Cauchy problem for the heat equation

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x \in (-\infty, +\infty), \quad t > 0, \\ u(x, 0) = \phi(x), & x \in (-\infty, +\infty), \end{cases} \tag{4}$$

where

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Show that u decays as $t \to \infty$ and find the decay rate. Explain, physically, why u decays as $t \to \infty$.

Solution. The bounded solution to the Cauchy problem is given by

$$u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi)\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)\mathrm{d}\xi = \frac{1}{2a\sqrt{\pi t}}\int_{-1}^{1}\phi(\xi)\exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)\mathrm{d}\xi.$$

Clearly, with an arbitrary fixed finite x,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\frac{1}{2a\sqrt{\pi t}}\int_{-1}^{1}\exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)\mathrm{d}\xi < \lim_{t\to\infty}\frac{1}{2a\sqrt{\pi t}}\int_{-1}^{1}\exp(0)\,\mathrm{d}\xi = 0,$$

Namely, u decays as $t \to \infty$. The (polynomial) decay rate as $t \to \infty$ is $1/2$ given by the part $\dfrac{1}{2a\sqrt{\pi t}} \sim \dfrac{1}{t^{1/2}}$. Physically, this means that finite initial thermal energy between $[-1, 1]$ needs to spread to infinity far, thus bringing a distribution decaying to 0 in the whole space as $t \to \infty$.

8 (Symmetry of heat equation)

Let u be the bounded solution of the Cauchy problem **7.(4)** with a general initial value. Show that if the initial value ϕ is even, then so is u in x; likewise, if ϕ is odd, then so is u in x. Hint: either use the explicit solution formula or use the maximum principle for the Cauchy problem.

Proof. Consider the explicit solution formula

$$u(x,t)=\int_{-\infty}^{+\infty}\phi(\xi)G(x,t;\xi)\mathrm{d}\xi=\frac{1}{2a\sqrt{\pi t}}\int_{-1}^{1}\phi(\xi)\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right)\mathrm{d}\xi,$$

where $G(x,t;\xi)=\dfrac{1}{2a\sqrt{\pi t}}\exp\left(-\dfrac{(x-\xi)^2}{4a^2t}\right)$ and it is not hard to see that

$$\int_{-\infty}^{+\infty}\phi(\xi)G(x,t;\xi)\mathrm{d}\xi=\int_{-\infty}^{+\infty}\phi(\xi)G(x,t;-\xi)\mathrm{d}\xi$$

for arbitrary ϕ.
If the initial ϕ is even, then with $\phi(\xi)G(-x,t;\xi)=\phi(-\xi)G(x,t;-\xi)=\phi(\xi)G(x,t;-\xi)$,

$$u(-x,t)=\int_{-\infty}^{+\infty}\phi(\xi)G(-x,t;\xi)\mathrm{d}\xi=\int_{-\infty}^{+\infty}\phi(\xi)G(x,t;-\xi)\mathrm{d}\xi=u(x,t)$$

for all t. Thus, $u(x,t)$ is also even in x.
If the initial ϕ is odd, then with $\phi(\xi)G(-x,t;\xi)=\phi(-\xi)G(x,t;-\xi)=-\phi(\xi)G(x,t;-\xi)$,

$$u(-x,t)=\int_{-\infty}^{+\infty}\phi(\xi)G(-x,t;\xi)\mathrm{d}\xi=-\int_{-\infty}^{+\infty}\phi(\xi)G(x,t;-\xi)\mathrm{d}\xi=-u(x,t)$$

for all t. Thus, $u(x,t)$ is also odd in x.

∎

9 (Black-Scholes equation)

Consider the terminal value problem for the Black-Scholes equation

$$\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2V}{\partial S^2}+rS\dfrac{\partial V}{\partial S}-rV=0, & S>0,\quad 0<t<T,\\ V(S,T)=\phi(S), & S>0,\end{cases}\tag{5}$$

where S is the price of a stock (as independent variable), V the call option value (as the dependent variable), σ the volatility of the stock, r the risk-free interest rate, T the expiration day of the option. This is designed to show that this terminal value problem can be transformed to the Cauchy problem for the heat equation and therefore (5) can be solved explicitly.

(1)

Introduce new variables

$$S = Ke^x, \quad t = T - \tau/(\sigma^2/2),$$

where the constant K is the striking price. Let $v(x,\tau) = V(S,t)$. Show that

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v.$$

Solution. Since

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV$$

and

$$S = Ke^x, t = T - \frac{2}{\sigma^2}\tau \quad \Rightarrow x = \ln\left(\frac{S}{K}\right),\ \tau = \frac{(T-t)\sigma^2}{2},$$

there are

$$\begin{aligned}
\frac{\partial V}{\partial S} &= \frac{\partial v}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}, \\
\frac{\partial V}{\partial t} &= \frac{\partial v}{\partial t} = \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}, \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial x}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) - \frac{1}{S^2}\frac{\partial v}{\partial x} = \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{S^2}\frac{\partial v}{\partial x}.
\end{aligned}$$

Thus

$$\begin{aligned}
-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} &= -\frac{1}{2}\sigma^2 S^2\left(\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{S^2}\frac{\partial v}{\partial x}\right) - rS\frac{1}{S}\frac{\partial v}{\partial x} + rv \\
\Rightarrow -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} &= -\frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma^2\frac{\partial v}{\partial x} - r\frac{\partial v}{\partial x} + rv \\
\Rightarrow \frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v.
\end{aligned}$$

(2)

Choose constants α and β such that

$$u(x,\tau) = \exp(\alpha x + \beta\tau)v(x,\tau)$$

satisfies the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Solution. The PDE for $u(x,\tau)$ can be given by

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu$$

where we want $A = B = 0$ by choosing α and β. With $v(x,\tau) = u(x,\tau)\exp(-\alpha x - \beta\tau)$,

$$\frac{\partial v}{\partial \tau} = \frac{\partial u \exp(-\alpha x - \beta\tau)}{\partial \tau}$$

$$
\begin{aligned}
&= \exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial \tau} + u\frac{\partial \exp(-\alpha x - \beta\tau)}{\partial \tau} \\
&= \exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial \tau} + u(-\beta)\exp(-\alpha x - \beta\tau), \\
\frac{\partial v}{\partial x} &= \frac{\partial u \exp(-\alpha x - \beta\tau)}{\partial x} \\
&= \exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u\frac{\partial \exp(-\alpha x - \beta\tau)}{\partial x} \\
&= \exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u(-\alpha)\exp(-\alpha x - \beta\tau), \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u(-\alpha)\exp(-\alpha x - \beta\tau)\right) \\
&= (-\alpha)\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + \exp(-\alpha x - \beta\tau)\frac{\partial^2 u}{\partial x^2} \\
&+ (-\alpha)\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u(-\alpha)^2\exp(-\alpha x - \beta\tau).
\end{aligned}
$$

Then, by substituting,

$$
\begin{aligned}
&\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial \tau} + u(-\beta)\exp(-\alpha x - \beta\tau) \\
&= (-\alpha)\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} \\
&+ \exp(-\alpha x - \beta\tau)\frac{\partial^2 u}{\partial x^2} + (-\alpha)\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u(-\alpha)^2\exp(-\alpha x - \beta\tau) \\
&+ \left(\frac{2r}{\sigma^2} - 1\right)\left(\exp(-\alpha x - \beta\tau)\frac{\partial u}{\partial x} + u(-\alpha)\exp(-\alpha x - \beta\tau)\right) - \frac{2r}{\sigma^2}u\exp(-\alpha x - \beta\tau)
\end{aligned}
$$

then dividing $\exp(-\alpha x - \beta\tau)$

$$
\frac{\partial u}{\partial \tau} + u(-\beta) = (-\alpha)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (-\alpha)\frac{\partial u}{\partial x} + u(-\alpha)^2 + \left(\frac{2r}{\sigma^2} - 1\right)\left(\frac{\partial u}{\partial x} + u(-\alpha)\right) - \frac{2r}{\sigma^2}u
$$

$$
\Rightarrow \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu
$$

where

$$
A = -2\alpha + \frac{2r}{\sigma^2} - 1,
$$

$$
B = \beta + \alpha^2 - \alpha\frac{2r}{\sigma^2} + \alpha - \frac{2r}{\sigma^2}.
$$

Set

$$
-2\alpha + \frac{2r}{\sigma^2} - 1 = 0 \quad \Rightarrow \alpha = \frac{r}{\sigma^2} - 1/2,
$$

$$
\beta + \left(\frac{r}{\sigma^2} - 1/2\right)^2 - \left(\frac{r}{\sigma^2} - 1/2\right)\frac{2r}{\sigma^2} + \left(\frac{r}{\sigma^2} - 1/2\right) - \frac{2r}{\sigma^2} = 0 \quad \Rightarrow \beta = \frac{r^2}{\sigma^4} + \frac{r}{\sigma^2} + \frac{1}{4},
$$

s.t u satisfies the heat equation.

(3)

Now solve (5) with $\phi(S) = \max(S - K, 0)$ (European call). Express your answer in terms of the distribution function of the normal distribution

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\xi^2/2} \mathrm{d}\xi.$$

Solution.

$$u(x,t) = \int_{-\infty}^{+\infty} \Phi(\xi) G(x,t;\xi) \mathrm{d}\xi = \int_{-\infty}^{+\infty} \exp(\alpha x) \max(K e^x - K, 0) G(x,t;\xi) \mathrm{d}\xi$$

The solution is

$$u(x,t) = \exp\left(\alpha x + \frac{1}{2}\sigma^2 \tau \alpha^2\right) N\left(\frac{x + \sigma^2 \tau \alpha}{\sigma\sqrt{\tau}}\right)$$

where

$$\mathrm{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\xi^2/2} \mathrm{d}\xi$$

is the cumulative distribution function of normalized normal distribution $N(0,1)$. Finally, we can perform backward substitutions from $u(x,\tau)$ to $V(S,t)$:

$$V(S,t) = S\mathrm{N}(d_1) - K e^{-r(T-t)} \mathrm{N}(d_2)$$

where

$$d_1 = \frac{1}{\sqrt{\sigma^2(T-t)}} \ln\left(\frac{S e^{\left(r + \frac{1}{2}\sigma^2\right)(T-t)}}{K}\right), \quad d_2 = \frac{1}{\sqrt{\sigma^2(T-t)}} \ln\left(\frac{S e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t)}}{K}\right).$$

10 (Application of maximum principles)

Let u be a smooth solution of the initial-boundary value problem

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x \in (0,l), \quad t > 0, \\ u(0,t) = 0 = u(l,t), & t > 0, \\ u(x,0) = \phi(x), & x \in [0,l], \end{cases} \tag{6}$$

where $\phi \geq 0$ but is not identically equal to zero on $[0,l]$, satisfying $\phi'' < 0$ on $(0,l)$.

(1) Prove that $u(x,t) > 0$ for $(x,t) \in (0,l) \times (0,\infty)$.

Hint: first use the weak minimum principle and then the strong minimum principle.

Proof. Denoting the parabolic interior $D_T = (0,l) \times (0,\infty) = \Omega \times (0,\infty)$ and the parabolic boundary $\Gamma_T = (\partial\Omega \times [0,\infty)) \cap (\overline{\Omega} \times \{0\})$, on D_T $u(x,t)$ satisfies

$$u_t \leq a^2 \Delta u.$$

According to the weak minimum principle,

$$\min_{\overline{D_T}} u = \min_{\Gamma_T} u = u(0,t) = 0.$$

Then, with the strong minimum principle, once the $u(x,t)$ value takes the minimum 0 on D_T, then u is a constant function, which is a contradiction. That is, $u(x,t) > 0$ for $(x,t) \in D_T$.

∎

(2) Prove that $u_x(x,t) < 0$ for $(x,t) \in (0,l) \times (0,\infty)$.

Hint: Let $w = u_x$; first find the initial-boundary problem that w solves, then apply the maximum principles to w.

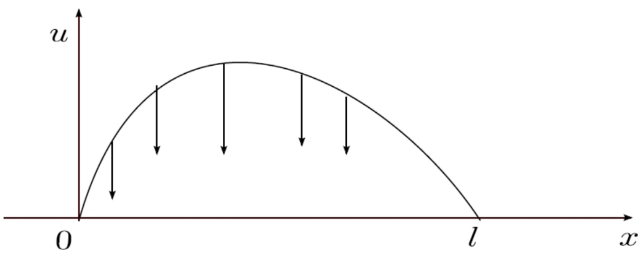
Proof. Let $w = u_t$, then

$$w = a^2 u_{xx} \quad \Rightarrow w_t = a^2 w_{xx}.$$

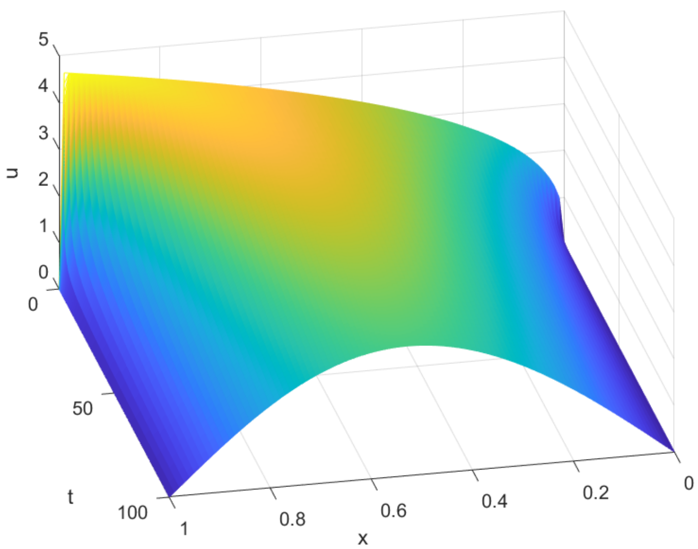
According to the strong maximum principle, with $w(x,0) = a^2 u_{xx}(x,0) = a^2\phi'' < 0$ on $(0,l)$ ($\phi'' \le 0$ on $[0,l]$ maybe), once the w value takes the maximum 0 on D_T, then w is a constant function, which is a contradiction. That is, $w(x,t) < 0$ for $(x,t) \in D_T$. ■

(3) Draw the graph of u vs x and put arrows on the graph to indicate the behavior of the graph as t increases...

Draw the graph of u vs x and put arrows on the graph to indicate the behavior of the graph as t increases. Can you predict the behavior without proving **(2)** rigorously? What if the initial value changes its concavity?

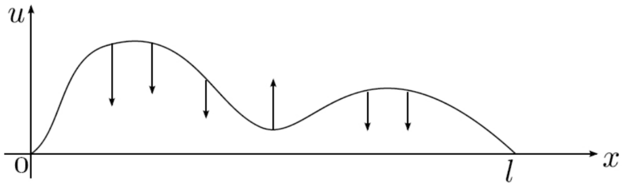


The behavior of u as t increases if the initial value is concave down

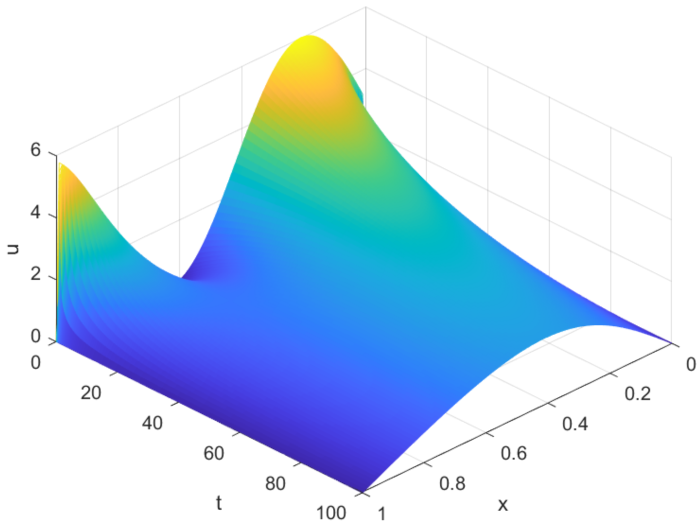


A Numerical Simulation of u, $\phi'' < 0$ on $(0,l)$

For each fixed x, as t increases u decrease, that is, $u_t < 0$, which can be predicted rigorously by the graph. If the initial value changes its concavity, i.e $\phi'' \geq 0$ may appear, the result $u_t < 0$ does not hold.



The behavior of u as t increases if the initial value changes its concavity



A Numerical Simulation of u, $\phi'' \geq 0$ on Somewhere $(0, l)$

11 Consider the solution $u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi) G(x,t;\xi) \mathrm{d}\xi$ of the Cauchy problem for the heat equation...

Consider the solution $u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi) G(x,t;\xi) \mathrm{d}\xi$ of the Cauchy problem for the heat equation. If ϕ is bounded on $\mathbb{R}$ and has a jump discontinuity at point x, prove that

$$\lim_{t \to 0^+} u(x,t) = \frac{1}{2} \left(\phi(x-0) + \phi(x+0) \right).$$

Proof. Let

$$u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi) G(x,t;\xi) \mathrm{d}\xi = \int_{-\infty}^{+\infty} \phi^-(\xi) G(x,t;\xi) \mathrm{d}\xi + \int_{-\infty}^{+\infty} \phi^+(\xi) G(x,t;\xi) \mathrm{d}\xi$$

where $\phi = \phi^- + \phi^+$,

$$\phi^-(\xi) = \begin{cases} \phi(\xi), & \xi \leq x - \epsilon \\ 0, & \xi > x \end{cases}, \quad \text{and } \phi^+(\xi) = \begin{cases} 0, & \xi < x \\ \phi(\xi), & \xi \geq x + \epsilon \end{cases},$$

as $\epsilon \to 0$, i.e. approaching x in the directions of 0^- or 0^+ respectively. Then, applied the weak convergence of fundamental solution,

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} \phi^-(\xi) G(x,t;\xi) \mathrm{d}\xi = \phi^-(x - \epsilon) = \phi^-(x - 0),$$

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} \phi^+(\xi) G(x,t;\xi) \mathrm{d}\xi = \phi^+(x + \epsilon) = \phi^-(x + 0).$$

Hence

$$\lim_{t \to 0^+} u(x,t) = \frac{1}{2}(\phi(x - 0) + \phi(x + 0)).$$

■

12 (Backward uniqueness of solving the heat equation)

We have already known the ill-posedness of solving the heat equation backwards in time. But, perhaps surprisingly, the backward heat equation has uniqueness, as we will prove in this exercise.
Let u be a smooth solution of

$$\begin{cases} u_t = a^2 u_{xx}, & (0 < x < l,\ t < 0), \\ u(0,t) = 0,\ u(l,t) = 0, & (t < 0), \\ u(x,0) = 0, & (0 < x < l). \end{cases}$$

Prove that u is identically equal to zero on $[0,l]$ for all $t \leq 0$. To this end, recall the energy that we have defined before

$$E(t) = \int_0^l u^2(x,t) \mathrm{d}x.$$

We just need to prove that $E(t)$ is identically equal to zero for all $t < 0$. We argue by contradiction by assuming that there exists $t_0 < 0$ such that $E(t_0) > 0$. By continuity, there exists a $t_1 \in (t_0, 0)$ such that E is positive on $[t_0, t_1]$ and is equal to zero at t_1. Without loss of generality, assume $t_1 = 0$. Now proceed as follows

(1)

Prove

$$E''(t) = 4a^2 \int_0^l u_{xx}^2(x,t) \mathrm{d}x.$$

Proof.

$$E'(t) = \frac{\mathrm{d}\left(\int_0^l u^2(x,t)\mathrm{d}x\right)}{\mathrm{d}t} = \int_0^l u u_t + u_t u \mathrm{d}x = 2\int_0^l u u_t \mathrm{d}x = 2a^2 \int_0^l u u_{xx} \mathrm{d}x.$$

Then

$$\begin{aligned}
E''(t) &= \frac{\mathrm{d}\left(2a^2\displaystyle\int_0^l uu_{xx}\mathrm{d}x\right)}{\mathrm{d}t} \\
&= 2a^2\int_0^l u_t u_{xx} + uu_{xxt}\mathrm{d}x \\
&= 2a^4\int_0^l u_{xx}^2\mathrm{d}x + 2a^2\int_0^l uu_{xxt}\mathrm{d}x \\
&= 2a^4\int_0^l u_{xx}^2\mathrm{d}x + uu_{xt}\left.\right|_{x=0}^l - 2a^2\int_0^l u_x u_{xt}\mathrm{d}x \\
&= 2a^4\int_0^l u_{xx}^2\mathrm{d}x + uu_{xt}\left.\right|_{x=0}^l - u_x u_t\left.\right|_{x=0}^l + 2a^2\int_0^l u_{xx}u_t\mathrm{d}x \\
&= 2a^4\int_0^l u_{xx}^2\mathrm{d}x + 2a^4\int_0^l u_{xx}^2\mathrm{d}x \\
&= 4a^4\int_0^l u_{xx}^2\mathrm{d}x
\end{aligned}$$

where $uu_{xt}\left.\right|_{x=0}^l = 0$ and $u_x u_t\left.\right|_{x=0}^l = 0$ are given by the B.C..

■

(2)

Prove Cauchy-Schwarz inequality

$$\left|\int_0^l f(x)g(x)\mathrm{d}x\right| \le \left(\int_0^l f^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^l g^2(x)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Hint: the quadratic polynomial in r defined by

$$\int_0^l (f(x)+rg(x))^2\mathrm{d}x$$

is always non-negative for all r; think about its discriminant.

Proof. Let

$$P(r) = \int_0^l \left(f(x)+rg(x)\right)^2\mathrm{d}x$$

then

$$\forall\, r,\ P(r) \ge 0.$$

Noticed that $P(r) = \displaystyle\int_0^l (f(x))^2\mathrm{d}x + 2r\int_0^l f(x)g(x)\mathrm{d}x + r^2\int_0^l (g(x))^2\mathrm{d}x$ is a quadratic function with $\displaystyle\int_0^l (g(x))^2\mathrm{d}x \ge 0$ s.t.

$$\Delta = \left(2\int_0^l f(x)g(x)\mathrm{d}x\right)^2 - 4\int_0^l (g(x))^2\mathrm{d}x\int_0^l (f(x))^2\mathrm{d}x \le 0.$$

That is,

$$\left(\int_0^l f(x)g(x)\mathrm{d}x\right)^2 \le \int_0^l (g(x))^2\,\mathrm{d}x \int_0^l (f(x))^2\,\mathrm{d}x,$$

or in a form of

$$\left|\int_0^l f(x)g(x)\mathrm{d}x\right| \le \sqrt{\int_0^l g^2(x)\mathrm{d}x}\sqrt{\int_0^l f^2(x)\mathrm{d}x}.$$

■

(3)

Use the formula for E' and E'' to prove

$$(E')^2 \le EE'', \quad t \in [t_0, 0).$$

Proof.

$$(E')^2 = \left(2a^2 \int_0^l uu_{xx}\mathrm{d}x\right)^2 = 4a^4\left(\int_0^l uu_{xx}\mathrm{d}x\right)^2 \le 4a^4 \int_0^l u^2\mathrm{d}x \cdot \int_0^l u_{xx}^2\mathrm{d}x = EE''.$$

■

(4)

Prove that

$$(\ln E(t))'' \ge 0, \quad t \in [t_0, 0).$$

Proof.

$$(\ln E(t))'' = \left(\frac{E'}{E}\right)' = \frac{E''E - (E')^2}{E^2} \ge 0$$

since $(E')^2 \le E''E$.

■

(5)

Prove that **(4)** contradicts the assumption that $u(x, 0) \equiv 0$ ($E(0) = 0$).

Proof. Suppose exist $t_0 < 0$ s.t. E is positive on $[t_0, 0)$ and $u(x, 0) \equiv 0$ ($E(0) = 0$).
By $(\ln E(t))'' \ge 0$, $\ln E(t)$ is a convex function. For $k \in [0, 1]$, with the property of convexity,

$$\ln\left(E\left(kt_0 + (1-k)\cdot 0\right)\right) \le k\ln(E(t_0)) + (1-k)\ln(E(0)) = \ln\left(E^k(t_0)E^{1-k}(0)\right)$$

$$\Rightarrow E\left(kt_0\right) \le E^k(t_0)E^{1-k}(0) = 0,$$

which is a contradiction.

■