PDE Introductory Exercises and Solutions
Chapter 2, First-order Partial Differential Equations

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Contents

1 Solve the following initial value problem: $3u_t + 5u_x = 0,\ u(x,0) = \exp(-x^2)$.

Solution. Let $\tau = 3t + 5x$, $\xi = 5t - 3x$, then

$$\begin{cases} u_t = \dfrac{\partial \tau}{\partial t}\dfrac{\partial u}{\partial \tau} + \dfrac{\partial \xi}{\partial t}\dfrac{\partial u}{\partial \xi} = 3u_\tau + 5u_\xi, \\ u_x = \dfrac{\partial \tau}{\partial x}\dfrac{\partial u}{\partial \tau} + \dfrac{\partial \xi}{\partial x}\dfrac{\partial u}{\partial \xi} = 5u_\tau - 3u_\xi. \end{cases}$$

We have the equation with new variables $3u_t + 5u_x = 34u_\tau = 0$, that is, $u_\tau = 0$. Thus, the general solution is given by

$$u(x,t) = f(\xi) = f(5t - 3x)$$

where f is a smooth function. With $t = 0$, $f(-3x) = \exp(-x^2)$, which implies $f(x) = \exp\left(-\dfrac{x^2}{9}\right)$. Hence,

$$u(x,t) = \exp\left(-\frac{(5t-3x)^2}{9}\right) = \exp\left(-\frac{25t^2 - 30xt + 9x^2}{9}\right).$$

2 Find the general solution of $u_x + xu_y = u$.

Solution. For the characteristic curves, we have

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}u}{u}.$$

Then

$$\begin{aligned} x\mathrm{d}x = \mathrm{d}y \quad &\Rightarrow \frac{x^2}{2} = y + c_1 \quad \Rightarrow \frac{x^2}{2} - y = c_1, \\ \mathrm{d}x = \frac{\mathrm{d}u}{u} \quad &\Rightarrow x = \ln u + c_2' \quad \Rightarrow e^{\ln u - x} = e^{-c_2'} \quad \Rightarrow ue^{-x} = c_2, \end{aligned}$$

where c_1, c_2' are arbitrary constants and denote $c_2 = e^{-c_2'}$. Thus, by $f(c_1) = c_2$ where f is an arbitrary smooth function,

$$f\left(\frac{x^2}{2} - y\right) = ue^{-x} \quad \Rightarrow u(x,y) = e^x f\left(\frac{x^2}{2} - y\right)$$

is a general solution.

3 Solve the initial value problem: $u_t + u_x = x,\ u(x,0) = 1/(1+x^2)$.

Solution. Let $\tau = t + x$, $\xi = t - x$, then

$$\begin{cases} u_t = u_\tau + u_\xi, \\ u_x = u_\tau - u_\xi. \end{cases}$$

We have the equation with new variables $u_t + u_x = 2u_\tau = x$. Thus, the general solution is given by

$$u(x,t) = \frac{x^2}{2} + f(t-x)$$

where f is a smooth function. With $t=0$, $\frac{x^2}{2} + f(-x) = \frac{1}{1+x^2}$, which implies $f(x) = -\frac{x^2}{2} + \frac{1}{1+x^2}$. Hence,

$$u(x,t) = \frac{x^2}{2} - \frac{(t-x)^2}{2} + \frac{1}{1+(t-x)^2} = -\frac{t^2}{2} + xt + \frac{1}{1+(t-x)^2}.$$

4 This exercise makes the point that the boundary condition for transport equations has to be given carefully...

This exercise makes the point that the boundary condition for transport equations has to be given carefully: show that the PDE $u_t + u_x = 0,\ x \in [0,1],\ t \in \mathbb{R}$, has no smooth solutions satisfying the boundary condition $u(0,t) = 1,\ u(1,t) = 2$. Explain this physically. Hint: Draw several characteristic curves.

Solution. Noted that $x = t$ is a characteristic curve, the general solution is given by $u(x,t) = f(t-x)$. With the boundary condition $u(0,t) = 1, u(1,t) = 2$,

$$f(t) = 1, \quad f(t-1) = 2,$$

and we see that $f(t)$ is a constant function. However, $f(t-1) = 2 \Rightarrow f(t) = 2$ implies that $f(t)$ has two different constant values, which is impossible for a smooth solution.

Physically, it means that at any time the mass at two end points of interval $[0,1]$ keep different constant velocities 1 and 2, which violates conservation of mass for transport equations.

5 Consider the following initial value problem for Burger's equation...

Consider the following initial value problem for Burger's equation

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = & \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

(1) Find the time t_s when shock first occurs;

For the characteristic equations, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 0,$$

then

$$x = ut + d, u = c.$$

With $\phi(d) = c(u) = u$, the family of characteristics parameterized by d is

$$x = \phi(d)t + c.$$

Thus, the general solution is given by

$$u(x,t) = \phi(d) = \phi(x - ut), \quad d = x - \phi(d)t.$$

It is an implicit relation that determines the solution of Burger's equation. For the time t_s when shock first occurs,

$$t_s = \inf_{x_0 \in \mathbb{R}, : \phi'(x_0) < 0} \left(-\frac{1}{c'(\phi(x_0))\, \phi'(x_0)} \right) = -\frac{1}{-1} = 1.$$

(2) Solve the initial value problem before time t_s.

$$u(x,t) = \phi(x - ut) = \begin{cases} 1, & x - ut \le 0, \\ 1 - (x - ut), & 0 < x - ut \le 1, \\ 0, & x - ut > 1. \end{cases}$$

then for $t < t_s$,

$$u(x,t) = \begin{cases} 1, & x \le t, \\ \dfrac{1-x}{1-t}, & t < x \le 1, \\ 0, & x > 1. \end{cases}$$

6 Let u be a positive C^1-smooth solution of Burger's equation $u_t + uu_x = 0,\ x \in (-\infty, \infty),\ t \ge 0$. Prove that...

Let u be a positive C^1-smooth solution of Burger's equation

$$u_t + uu_x = 0, \quad x \in (-\infty, \infty),\ t \ge 0.$$

Prove that (1) for each fixed $t \ge 0$, u is non-decreasing in x; (2) for each fixed x, u is non-increasing in $t \ge 0$. Hint: Argue by contradiction to prove (1).

(1) for each fixed $t \ge 0$, u is non-decreasing in x;

Proof. In fact, a monotonically decreasing continuous solution in x for Burger's equation does not exist. But the following argument proves the result directly.

With t fixed and $u = \phi(x - ut)$,

$$\frac{\partial u}{\partial x} = \frac{\partial \phi(x-ut)}{\partial x} = \frac{\partial \phi(x-ut)}{\partial (x-ut)} \frac{\partial (x-ut)}{\partial x} = \phi' \cdot \left(\frac{\partial x}{\partial x} - \frac{\partial (ut)}{\partial x} \right) = \phi' \cdot \left(1 - \frac{t\partial u}{\partial x} \right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\phi'}{1 + t\phi'}$$

where $t < t_s$ already promises $1 + t\phi' \neq 0$.

If for all x, $\dfrac{\partial u}{\partial x} = \dfrac{\phi'}{1 + t\phi'} \leq 0$, then it requires $\phi'(1 + t\phi') = t\phi'^2 + \phi' < 0$. However, for this quadratic function of ϕ' (the two zero points are $\phi' = 0, \phi' = -1/t$), it is a contradiction since $t > 0$.

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(2) for each fixed x, u is non-increasing in $t \geq 0$.

Proof. In fact, a monotonically increasing continuous solution in t for Burger's equation does not exist. But the following argument proves the result directly.

With x fixed and $u = \phi(x - ut)$,

$$\frac{\partial u}{\partial t} = \frac{\partial \phi(x - ut)}{\partial t} = \frac{\partial \phi(x - ut)}{\partial (x - ut)} \frac{\partial (x - ut)}{\partial t} = \phi' \cdot \left(\frac{\partial x}{\partial t} - \frac{t \partial u}{\partial t} - \frac{u \partial t}{\partial t} \right) = \phi' \cdot \left(-\frac{t \partial u}{\partial t} - u \right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u\phi'}{1 + t\phi'}$$

where $t < t_s$ already promises $1 + t\phi' \neq 0$.

If for all t, $\dfrac{\partial u}{\partial t} = \dfrac{-u\phi'}{1 + t\phi'} \geq 0$, that is, by $u > 0$, $\dfrac{\partial u}{\partial t} = \dfrac{\phi'}{1 + t\phi'} \leq 0$, then it requires $\phi'(1 + t\phi') = t\phi'^2 + \phi' < 0$. However, for all these quadratic functions of ϕ' (the two zero points are $\phi' = 0, \phi' = -1/t$), it is a contradiction since every parameter $t > 0$.

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