PDE Introductory Exercises and Solutions
Chapter 1, Introduction

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Contents

1 For each of the PDEs below, find its order, linearity and homogeneity.

(1) $u_t + uu_x = 0$ **(Burger's equation).**

1st order, nonlinear, since $L[u+v] = (u+v)_t + (u+v)\cdot(u+v)_x = u_t + v_t + uu_x + uv_x + vu_x + vv_x \neq L[u] + L[v]$.

(2) $xu_t - u_{xx} + 2x + \sin t = 0$ **(Degenerate heat equation).**

2nd order, linear, inhomogeneous, since $L[u] = xu_t - u_{xx} = -2x - \sin t$.

(3) $u_{tt} - (u_{xx} + u_{yy} + u_{zz}) = -u + u^3$ **(Klein-Gordon equation).**

2nd order, nonlinear.

(4) $(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0$ **(Minimal surface equation).**

2nd order, nonlinear.

2 Classify the following equations as hyperbolic, parabolic or elliptic.

(1) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$.

Take $a_{11} = 1$, $a_{22} = 5$, $a_{12} = a_{21} = 2$. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = 1 > 0$. The PDE is elliptic.

(2) $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u = 0$.

Take $a_{11} = 1$, $a_{22} = 4$, $a_{12} = a_{21} = -2$. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = 0$. The PDE is parabolic.

(3) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0$.

Take $a_{11} = 1$, $a_{22} = -3$, $a_{12} = a_{21} = 1$. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = -4 < 0$. The PDE is hyperbolic.

3 Show that the only ones that are unchanged under all axis-rotations (rotation invariant)...

Consider the PDE with constant coefficients

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = 0.$$

Show that the only ones that are unchanged under all axis-rotations (rotation invariant) have the form

$$a \cdot (u_{xx} + u_{yy}) + bu = 0,$$

where a and b are constants.

Proof. The rotation with counterclockwise angle θ in xy-plane can be written as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and let

$$v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta x - \sin\theta y \\ \sin\theta x + \cos\theta y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Since

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos\theta\frac{\partial}{\partial x'} + \sin\theta\frac{\partial}{\partial y'}, \\
\frac{\partial}{\partial y} &= \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = -\sin\theta\frac{\partial}{\partial x'} + \cos\theta\frac{\partial}{\partial y'}, \\
\frac{\partial^2}{\partial x^2} &= \cos^2\theta\frac{\partial^2}{\partial x'^2} + 2\cos\theta\frac{\partial}{\partial x'}\sin\theta\frac{\partial}{\partial y'} + \sin^2\theta\frac{\partial^2}{\partial y'^2}, \\
\frac{\partial^2}{\partial y^2} &= \sin^2\theta\frac{\partial^2}{\partial x'^2} - 2\sin\theta\frac{\partial}{\partial x'}\cos\theta\frac{\partial}{\partial y'} + \cos^2\theta\frac{\partial^2}{\partial y'^2}, \\
\frac{\partial^2}{\partial xy} &= -\sin\theta\cos\theta\frac{\partial^2}{\partial x'^2} + \left(\cos^2\theta - \sin^2\theta\right)\frac{\partial^2}{\partial x'\partial y'} + \sin\theta\cos\theta\frac{\partial^2}{\partial y'^2},
\end{aligned}$$

it comes

$$\begin{aligned}
& a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cv \\
=& a_{11}\left(\cos^2\theta v_{xx} + 2\cos\theta\sin\theta v_{xy} + \sin^2\theta v_{yy}\right) \\
& + 2a_{12}\left(-\sin\theta\cos\theta v_{xx} + \left(\cos^2\theta - \sin^2\theta\right)v_{xy} + \sin\theta\cos\theta v_{yy}\right) \\
& + a_{22}\left(\sin^2\theta v_{xx} - 2\sin\theta\cos\theta v_{xy} + \cos^2\theta v_{yy}\right) \\
& + b_1\left(\cos\theta v_x + \sin\theta v_y\right) \\
& + b_2\left(-\sin\theta v_x + \cos\theta v_y\right) \\
& + cv \\
=& \left(a_{11}\cos^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta\right)v_{xx} \\
& + 2\left(2a_{11}\cos\theta\sin\theta + 2a_{12}\left(\cos^2\theta - \sin^2\theta\right) - 2a_{22}\sin\theta\cos\theta\right)v_{xy} \\
& + \left(a_{11}\sin^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right)v_{yy} \\
& + \left(b_1\cos\theta - b_2\sin\theta\right)v_x \\
& + \left(b_1\sin\theta + b_2\cos\theta\right)v_y \\
& + cv \\
=& a'_{11}v_{xx} + 2a'_{12}v_{xy} + a'_{22}v_{yy} + b'_1v_x + b'_2v_y + cv \\
=& 0.
\end{aligned}$$

To satisfy the rotation invariance, the coefficients should not be changed under all axis-rotations

$$a_{11} = a'_{11}, \quad a_{12} = a'_{12}, \quad a_{22} = a'_{22}, \quad b_1 = b'_1, \quad b_2 = b'_2,$$

i.e.

$$
\begin{aligned}
a_{11} &= \left(a_{11}\cos^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta\right), \\
a_{12} &= \left(2a_{11}\cos\theta\sin\theta + 2a_{12}\left(\cos^2\theta - \sin^2\theta\right) - 2a_{22}\sin\theta\cos\theta\right), \\
a_{22} &= \left(a_{11}\sin^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right), \\
b_1 &= (b_1\cos\theta - b_2\sin\theta), \\
b_2 &= (b_1\cos\theta - b_2\sin\theta).
\end{aligned} \tag{1}
$$

Noticed that there must not be interleaved coefficient of terms, with applying $\sin^2\theta + \cos^2\theta = 1$, a solution is given by

$$a_{11} = a_{22}, \quad a_{12} = 0, \quad b_1 = 0, \quad b_2 = 0, \tag{2}$$

which exactly means the form

$$a \cdot (u_{xx} + u_{yy}) + bu = 0. \tag{3}$$

Besides, it is not hard to see that (2) is the only possible solution of (1) s.t. the rotation invariant form (3) is unique.

■

4 Classify the following equations as hyperbolic, parabolic or elliptic...

Classify the following equations as hyperbolic, parabolic or elliptic. If the type changes in the xy-plane, find the region for each type.

(1) $(1+x^2)u_{xx} + (1+y^2)u_{yy} + xu_x + yu_y = 0.$

Take $a_{11} = (1+x^2)$, $a_{22} = (1+y^2)$, $a_{12} = a_{21} = 0$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = (1+x^2)(1+y^2).$$

Clearly, $(1+x^2)(1+y^2) > 0$ for all $x, y \in \mathbb{R}$. Thus, the PDE is elliptic.

(2) $u_{xx} + (1+y)^2 u_{yy} = 0.$

Take $a_{11} = 1$, $a_{22} = (1+y)^2$, $a_{12} = a_{21} = 0$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = (1+y)^2.$$

$y = -1, (1+y)^2 = 0$, that is, on the line $y = -1$ the PDE is parabolic; in the rest of the xy-plane $(1+y)^2 > 0$, it is elliptic.

(3) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0.$

Take $a_{11} = e^{2x}$, $a_{22} = e^{2y}$, $a_{12} = a_{21} = e^{x+y}$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = e^{2x+2y} - e^{2(x+y)} = 0.$$

The PDE is parabolic.

(4) $u_{xx} + yu_{yy} = 0.$

Take $a_{11} = 1$, $a_{22} = y$, $a_{12} = a_{21} = 0$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = y.$$

The PDE is elliptic on the region $y > 0$; it is parabolic on the line $y = 0$; it is hyperbolic on the region $y < 0$.

(5) $u_{xx} + xyu_{yy} = 0.$

Take $a_{11} = 1$, $a_{22} = xy$, $a_{12} = a_{21} = 0$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = xy.$$

The PDE is elliptic on the region $xy > 0$ (that is, the $x > 0, y > 0$ part and $x < 0, y < 0$ part of the xy-plane); it is parabolic on the region $xy = 0$ (that is, on the x-axis or y-axis); it is hyperbolic on the region $xy < 0$ (that is, the $x > 0, y < 0$ part and $x < 0, y > 0$ part of the xy-plane).

(6) $yu_{xx} - xu_{yy} + u_x + yu_y = 0.$

Take $a_{11} = y$, $a_{22} = -x$, $a_{12} = a_{21} = 0$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 = -xy.$$

The PDE is elliptic on the region $-xy > 0$ (that is, the $x > 0, y < 0$ part and $x < 0, y > 0$ part of the xy-plane); it is parabolic on the region $-xy = 0$ (that is, on the x-axis or y-axis); it is hyperbolic on the region $-xy < 0$ (that is, the $x > 0, y > 0$ part and $x < 0, y < 0$ part of the xy-plane).

5 Show by direct substitution that $u(x,t) = f(x+2t) + g(x-2t)$ is a solution of the PDE $u_{tt} - 4u_{xx} = 0$ ...

Show by direct substitution that $u(x,t) = f(x+2t) + g(x-2t)$ is a solution of the PDE

$$u_{tt} - 4u_{xx} = 0$$

for arbitrary smooth functions f and g.

Solution. Let $x + 2t = a, x - 2t = b$. We have

$$u_{tt} = \frac{\partial}{\partial t}\left(\frac{\partial f(x+2t)}{\partial(x+2t)}\frac{\partial(x+2t)}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial g(x-2t)}{\partial(x-2t)}\frac{\partial(x-2t)}{\partial t}\right) = 2\frac{\partial}{\partial t}\frac{\partial f}{\partial a} - 2\frac{\partial}{\partial t}\frac{\partial g}{\partial b},$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f(x+2t)}{\partial(x+2t)}\frac{\partial(x+2t)}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial g(x-2t)}{\partial(x-2t)}\frac{\partial(x-2t)}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial a} + \frac{\partial}{\partial x}\frac{\partial g}{\partial b}.$$

Also,

$$\frac{\partial}{\partial t} = \frac{\partial a}{\partial t}\frac{\partial}{\partial a} + \frac{\partial b}{\partial t}\frac{\partial}{\partial b} = 2\frac{\partial}{\partial a} - 2\frac{\partial}{\partial b},$$
$$\frac{\partial}{\partial x} = \frac{\partial a}{\partial x}\frac{\partial}{\partial a} + \frac{\partial b}{\partial x}\frac{\partial}{\partial b} = \frac{\partial}{\partial a} + \frac{\partial}{\partial b}.$$

Then,

$$\begin{aligned}
u_{tt} - 4u_{xx} &= 2\frac{\partial}{\partial t}\frac{\partial f}{\partial a} - 2\frac{\partial}{\partial t}\frac{\partial g}{\partial b} - 4\frac{\partial}{\partial x}\frac{\partial f}{\partial a} - 4\frac{\partial}{\partial x}\frac{\partial g}{\partial b} \\
&= 2\left(2\frac{\partial}{\partial a} - 2\frac{\partial}{\partial b}\right)\frac{\partial f}{\partial a} - 2\left(2\frac{\partial}{\partial a} - 2\frac{\partial}{\partial b}\right)\frac{\partial g}{\partial b} - 4\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)\frac{\partial f}{\partial a} - 4\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)\frac{\partial g}{\partial b} \\
&= 4\frac{\partial}{\partial a}\frac{\partial f}{\partial a} - 4\frac{\partial}{\partial b}\frac{\partial f}{\partial a} - 4\frac{\partial}{\partial a}\frac{\partial g}{\partial b} + 4\frac{\partial}{\partial b}\frac{\partial g}{\partial b} - 4\frac{\partial}{\partial a}\frac{\partial f}{\partial a} - 4\frac{\partial}{\partial b}\frac{\partial f}{\partial a} - 4\frac{\partial}{\partial a}\frac{\partial g}{\partial b} - 4\frac{\partial}{\partial b}\frac{\partial g}{\partial b} \\
&= -8\frac{\partial}{\partial b}\frac{\partial f(a)}{\partial a} - 8\frac{\partial}{\partial a}\frac{\partial g(b)}{\partial b} \\
&= -8\cdot 0 - 8\cdot 0 \\
&= 0.
\end{aligned}$$

Hence, $u(x,t) = f(x+2t) + g(x-2t)$ is a solution of the PDE.

(Another way to demonstrate this is by calculating u_t, u_x then u_{tt}, u_{xx} directly with $f'(x+2t), g'(x-2t), f''(x+2t), g''(x-2t)$.)

6 Find the general solution of the equation $Au_{xx} + Bu_{xy} + Cu_{yy} = 0$ $(*)$ where...

Given the partial differential equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} = 0, \qquad (*)$$

where A, B and C are constants. Find the general solution of the above equation when

(1) equation $(*)$ is hyperbolic;

Take $a_{11} = A$, $a_{22} = C$, $a_{12} = a_{21} = B/2$. And there is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = AC - \frac{B^2}{4} < 0 \quad \Rightarrow B^2 - 4AC > 0.$$

(a) If $A = 0$:
It requires $B \neq 0$, then $(*)$ becomes

$$\left(u_x + \frac{C}{B}u_y\right)_y = 0.$$

It follows that

$$u_x + \frac{C}{B}u_y = f(x).$$

The characteristics of this equation are given by

$$\frac{dy}{dx} = \frac{C}{B},$$

hence

$$y = \frac{C}{B}x + D.$$

Now on a fixed characteristic curve (so constant D is fixed), we have

$$\frac{du}{dx} = u_x + \frac{C}{B}u_y = f(x),$$

from which we derive

$$u = F(x) + \tilde{D},$$

where F is the antiderivative of f. The constant $\tilde{D}$ depends on the characteristic curve, and hence on D. Let $\tilde{D} = G(D)$ and solve for D, then the solutions are given by

$$u(x, y) = F(x) + G\left(y - \frac{C}{B}x\right).$$

(b) If $A \neq 0$:
For the 2nd-order linear PDE, we have characteristic equation $A(\mathrm{d}y)^2 + B(\mathrm{d}x)(\mathrm{d}y) + C(\mathrm{d}x)^2 = 0$. Rewrite as

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + B\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + C = 0,$$

and there are two real solutions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2.$$

Then, by integrating,

$$y - \lambda_1 x = c_1, \quad y - \lambda_2 x = c_2.$$

In terms of c_1 and c_2, the second-order PDE reduces to

$$\frac{\partial^2 u}{\partial c_1 \partial c_2} = 0,$$

for which the general solution is $u(c_1, c_2) = f(c_1) + g(c_2)$ s.t.

$$\begin{aligned} u(x, y) &= f(y - \lambda_1 x) + g(y - \lambda_2 x) \\ &= f\left(y - \frac{-B + \sqrt{B^2 - 4AC}}{2A}x\right) + g\left(y - \frac{-B - \sqrt{B^2 - 4AC}}{2A}x\right) \end{aligned}$$

where f, g are arbitrary smooth functions.

(2) equation (∗) is parabolic.

Take $a_{11} = A$, $a_{22} = C$, $a_{12} = a_{21} = B/2$. And there is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = AC - \frac{B^2}{4} = 0 \quad \Rightarrow B^2 - 4AC = 0.$$

(a) If $A = 0$:
It requires $B = 0$, then (∗) reduces to

$$u_{yy} = 0.$$

Obviously the general solution is

$$u(x, y) = f(x)y + g(x).$$

(b) If $A \neq 0$:
For the 2nd-order linear PDE, we have characteristic equation $A(\mathrm{d}y)^2 + B(\mathrm{d}x)(\mathrm{d}y) + C(\mathrm{d}x)^2 = 0$. Rewrite as

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + B\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + C = 0,$$

which comes with two equal real solutions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-B}{2A} = \lambda.$$

Then, the characteristic curves are given by

$$y - \lambda x = c_1.$$

Since the above only gives one characteristic direction, we introduce the following change of variables

$$c_1 = y - \lambda x, \quad c_2 = y + \lambda x.$$

In terms of c_1 and c_2, the second-order PDE reduces to

$$\frac{\partial^2 u}{\partial c_1^2} = 0,$$

for which the general solution is $u(c_1, c_2) = c_1 f(c_2) + g(c_2)$ s.t.

$$\begin{aligned} u(x, y) &= (y - \lambda x) f(y + \lambda x) + g(y + \lambda x) \\ &= \left(y + \frac{B}{2A}x\right) f\left(y - \frac{B}{2A}x\right) + g\left(y - \frac{B}{2A}x\right) \end{aligned}$$

where f, g are arbitrary smooth functions.